Experiment 6 - Time domain analysis of an RC circuit

Achievements in this experiment

You will analyse a simple network using tools such as steps, impulses, exponential pulses and sinusoids to compare theory and practical results. You will then synthesise and test an equivalent network using a 1st order feedback structure.

Preliminary discussion

The RC circuit is a simple form of network which involves electrical charge storage elements. It is these "storage elements" such as capacitors and inductances which create the delays in signals often referred to as "leads" and "lags" in signal phases relative to each other.

![RC Network Diagram]

**Figure 1: RC network; circuit and block diagram**

From the experiment on Exponentials, remember that RC is the time taken for an unit exponential to decay to \( e^{-1} = 1/e = 0.37 \) of its initial value
Pre-requisite work

Question 1: the step response

(a) Apply elementary circuit theory to show that the voltage equation for the RC circuit

\[ V_{in}(t) = i(t), R + V_{cap}(t) = i(t), R + Q(t)/C \quad \text{Eq prep1.1} \]

Where \( V_{cap}(t) \) is the voltage across the capacitor with capacitance \( C \) and \( Q(t) \) is the charge in capacitor.

(b) Show that this can be expressed as

\[ (d/dt)(V_{in}) = R \cdot di/dt + i/C \]

Consider the case where \( V_{in}(t) \) is a step function of amplitude \( V_o \) and the capacitor charge \( Q(t) = 0 \) at \( t = 0 \). Show that for \( t > 0 \) \((d/dt)(V_{in}) = 0\) and the DE reduces to

\[ di/dt = -a, i \quad [a = (1/RC)] \]

Use \((d/dt)\log_e(i) = 1/i\) to show that the solution of the DE is

\[ i(t) = i_o \exp(-a.t) \quad (t > 0) \quad [i_o = V_o/R] \]

(c) Use Eq.1.1 to show that

\[ V_{out}(t) = V_{cap}(t) = V_{in}(t) - R \cdot i_o \exp(-a.t) \]

Hence the step response \( V_{out}/V_{in} = (1 - \exp(-a.t)) \)

(d) Plot the result in (c) for \( a = 1000 \)

(e) What is the asymptotic value of the step response as \( t \) increases indefinitely? Show that the step response rises to \((1 - 1/e)\) of its final value at \( t = 1/a \).

Question 2: the impulse response

(a) Describe the main properties of the theoretical impulse function.

Show that differentiation of the unit step function \( \text{wrt } t \) produces a unit impulse at \( t = 0 \). Apply this to the step response result in Question P1(c) to show that the impulse response \( h(t) \) of the RC circuit is

\[ a \cdot \exp(-a.t) \]
(b) Explain why the impulse function can only be approximated in practice. Sketch an impulse approximation realized as a finite width pulse. Explain why an excessively narrow pulse is undesirable in practical applications. Estimate a pulse width that would be suitable for use with the case in Question P1. Indicate your reasoning.

(c) Using the property in (a) we could generate the impulse response by first recording the step response, then differentiating. Compare this alternative with the use of a finite width pulse input. Include discussion of signal peak limitations and output amplitude considerations.

(d) Show that the impulse response falls to 1/e of its initial value at 
\[ t = \frac{1}{a} \]

**Question 3: convolution and response to an exponential input**
This question introduces convolution and its application in the analysis of systems like the RC circuit in Q.P.1.

(a) The convolution of the time functions \( x_1 \) and \( x_2 \) can be expressed as

\[
x_1 * x_2 = \int_0^t x_1(\tau) x_2(t - \tau)d\tau \quad \text{[for } t > 0] \]

Note that the convolution is a function of \( t \) and that tau is a dummy variable that has no further role after integration.

Show that changing the order \((x_2 * x_1)\) does not change the result.

Show that if \( x_1 \) is a unit impulse the convolution \( x_1 * x_2 = x_2(t) \).

Suppose we approximate a continuous time signal \( x_1(t) \) as a sum of very narrow contiguous pulses, each of which can be thought of as representing an impulse function (each with its individual amplitude). Suppose next that this pulse train representation of \( x_1(t) \) is then applied as input to the system introduced in Q. P1. Each of the pulses in the train will produce an individual output that will be a close (weighted) approximation to the system’s impulse response. The overall output will be the sum of these (overlapping) weighted impulse response approximations.

Demonstrate that this sum is effectively the convolution of \( x_1 \) and the system’s impulse response \( h(t) \). (Invoke the usual limit methods to morph the discrete sum into a continuous time integral.)

(b) Show that for \( t > 0 \), the convolution for the case

\[
x_1(t) = \exp(-a_1 t) \text{ and } x_2(t) = \exp(-a_2 t) \quad \text{[a1 N.E. a2]}
\]

is \((1/(a_2 - a_1)) \cdot (\exp(-a_1 t) - \exp(-a_2 t))\)

(c) Sketch the graph of the result in (b) versus \( t \) over the range \( t > 0 \). Show that for positive values of \( a_1 \) and \( a_2 \) the function is positive for \( t > 0 \), and that it is zero at \( t = 0 \) and \( t \to \infty \). Find the peak and the corresponding value of \( t \) for \( a_1 = 0.5 \) and \( a_2 = 1.1 \).

(d) Use the results in (a) and (b) and in Q. P2(a) to obtain the response of the RC circuit in Q.P1 when the input is
\[ x_1(t) = \exp(-a_1 t) \]

(e) Repeat the tasks in (b) and (c) for the case \( a_1 = a_2 = a \).

NB: a useful reference for this question is Schuam Laplace Transforms (1965); p45 (convolution of two exponentials)

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**Question 4: response to a sinusoidal input**

In Q. P1 we sought the output of the RC circuit in Fig xxx for the case in which the input is a step function. This result was extended in Q. P2 and P3 for an impulse function input and for an exponential input. Now we examine the solution when the input is sinusoidal. This case is of special importance in this work as it opens the way to powerful tools for the solution of systems of much greater complexity than the introductory example under investigation here.

(a) Use the result in Q.P1(a) to show that

\[ V_{\text{in}}(t) = RC.(dV_{\text{out}}/dt) + V_{\text{out}}(t) \quad \text{Eqn. P4.1} \]

To simplify the analysis we will use the complex exponential \( A_{\text{in}} \exp(j\omega t) \) to represent the input sinusoid [recall that \( \exp(j\omega t) = \cos(\omega t) + j\sin(\omega t) \)].

In Q. P1(b) we obtained a solution of the DE by direct integration. However, sometimes it turns out that invoking a “feeling lucky” approach can provide the desired result:

a solution of the form

\[ V_{\text{out}} = A_{\text{out}} \cdot \exp(j\phi_{\text{out}}) \cdot \exp(j\omega t) \]

is substituted into the RHS in the above DE.

Show that this is a solution for a suitable value of \( A_{\text{out}} \cdot \exp(j\phi_{\text{out}}) \). (The suitable value is the one that makes the RHS = LHS). With \( A_{\text{in}} = 1 \), show that the sought value is

\[ A_{\text{out}} \cdot \exp(j\phi_{\text{out}}) = 1/(1 + j\omega RC) \]

Hence show that

\[ V_{\text{out}} = V_{\text{in}} \cdot 1/(1 + j\omega RC) = V_{\text{in}} \cdot (1/RC)/ (j\omega + (1/RC)) \]

Note that this result has a very interesting feature:

the output has the same form as the input.

[To discover the importance of this property it is worthwhile to think about the use of other waveforms to express the output in terms of the input. For example, a squarewave, a periodic ramp, a sawtooth (an optional lab exercise). ]

(b) Use the result in (a) to obtain a formula for the ratio of output amplitude to input amplitude as a function of \( \omega \) for \( 1/RC = 1000 \) (rad/sec). Sketch the result, and find the value of \( \omega \) for which the ratio is 3dB.

**Question 5: solution using the Laplace Transform**

(a) Look up the definition \( Y(s) \) of the Laplace transform of the
function \( y(t) \). Show that the Laplace transform of \( (d/dt)y(t) \) is \( sY(s) \).

Solve Eqn P4.1 as a function of \( s \) by applying the Laplace transform to both sides (note that no restriction is imposed on the form of the input).

Compare this result with the solution obtained with input

\[
V_{\text{in}}(t) = A_{\text{in}} \exp(j \omega t)
\]

Comment on similarities and differences.

(b) The transfer function is defined as \( V_{\text{out}}(s)/V_{\text{in}}(s) \). Use the result in (a) to write down the transfer function of the RC circuit.

(c) Find the Laplace transform of \( y(t) = \exp(-a \cdot t) \). Compare this with the transfer function in (b).

(d) What is the relationship between the transfer function and the impulse response that is apparent from (c)?

(e) On the basis of (d), what is the operation in the \( s \) domain that corresponds to convolution in the time domain? Confirm your answer by looking up the convolution theorem.

**Question 6: synthesized model of RC circuit**

(a) Consider Eqn P4.1 in the Laplace domain, i.e.,

\[
s.V_{\text{out}} = a . V_{\text{in}} + (-a) . V_{\text{out}}
\]

Use the block diagram in Task 25 as a guide to model this equation using an integrator (1/s). Note that \( s.V_{\text{out}}(s) \) appears at the integrator input.

(b) In practical applications the use of a scaled integrator \( (k/s) \) may be necessary. Adjust the system equation so that the LHS is \( (s/k).V_{\text{out}} \), and modify the model accordingly.

(c) Suppose \( k = 200 \) and \( a = 1000 \). Determine the corresponding value of \( a1 \) in the block diagram in Task 25.
Equipment

- PC with LabVIEW Runtime Engine software appropriate for the version being used.
- NI ELVIS 2 or 2+ and USB cable to suit
- EMONA SIGEx Signal & Systems add-on board
- Assorted patch leads
- Two BNC - 2mm leads

Procedure

Part A - Setting up the NI ELVIS/SIGEx bundle

1. Turn off the NI ELVIS unit and its Prototyping Board switch.
2. Plug the SIGEx board into the NI ELVIS unit.

Note: This may already have been done for you.

3. Connect the NI ELVIS to the PC using the USB cable.
4. Turn on the PC (if not on already) and wait for it to fully boot up (so that it’s ready to connect to external USB devices).
5. Turn on the NI ELVIS unit but not the Prototyping Board switch yet. You should observe the USB light turn on (top right corner of ELVIS unit). The PC may make a sound to indicate that the ELVIS unit has been detected if the speakers are activated.
6. Turn on the NI ELVIS Prototyping Board switch to power the SIGEx board. Check that all three power LEDs are on. If not call the instructor for assistance.
7. Launch the SIGEx Main VI.
8. When you’re asked to select a device number, enter the number that corresponds with the NI ELVIS that you’re using.
9. You’re now ready to work with the NI ELVIS/SIGEx bundle.
10. Select the Lab 10 tab on the SIGEx SFP.

Note: To stop the SIGEx VI when you’ve finished the experiment, it’s preferable to use the STOP button on the SIGEx SFP itself rather than the LabVIEW window STOP button at the top of the window. This will allow the program to conduct an orderly shutdown and close the various DAQmx channels it has opened.
Experiment

Part 1: Step response of the RC network

In previous experiments you have been introduced to the step response as a useful signal with which to investigate a system.

In this part of the experiment we will measure the step response of an RC network and compare it to our theoretical expectations.

![Image of RC network and Pulse Generator block diagram]

**Figure 2**: block diagram of RC network; wiring diagram of experiment

11. Wire together the RC NETWORK block with the PULSE GENERATOR block as the source of input signal.

Settings are as follows:
- **PULSE/CLK GENERATOR**: 50 Hz; DUTY CYCLE=0.5 (50%)
- **SCOPE**: Timebase 20ms; Rising edge trigger on CH1; Trigger level=1V
- Connect CH1 to input, CH0 to output

From the preliminary discussion at the beginning of this lab, the unit step response of an RC network is:

\[ h(t) = \left[1 - e^{-t/RC}\right]u(t) \]

12. Drawing upon your pre-lab preparation work, use the actual values of the SIGEx RC network to calculate the expected step response signal.

Use the values: \( R = 10,000 \text{ ohm}, \ C = 100nF \ (100 \times 10^{-9} \text{ F}) \); hence \( RC = 1 \times 10^{-3} \), and \( 1/RC = 10^3 = 1000 \). The time constant for this circuit is 1 ms.

Measure the input step size in volts.

**Question 7**

How long will it take this RC NETWORK to rise to a level 37% below its final level?
Question 8
Calculate the expected real circuit step response of the RC NETWORK using the real circuit values and real circuit input values. These values are available in the User Manual. For your convenience they are R=10kohm, and C=100nF

13. Confirm that the measured step response corresponds with your theoretical expectations. Characteristics of the exponential waveform were discussed in Experiment 7 on exponentials.

![Graph of exponential waveform](image)

Figure 3: exponential waveform $Ae^{(-t)}u(t)$; where $u(t)$ is Heaviside function
14. Sketch the step response on the graph below. Show all relevant time constants and voltage levels.

Graph 1: step and impulse responses

**Impulse response of the RC network**

In earlier experiments where we introduced the impulse function, it was noted that a true impulse, with infinitesimally small width, and infinite amplitude, cannot be actually physically generated. It can however be approximated, by a pulse of finite height and nonzero duration. Our criterion for whether the approximation is adequate is that the impulse is sufficiently brief that there is no discernible change in the “system under investigation’s” output shape as the pulse width is reduced. This is the criteria we will apply here in creating an impulse.

15. Maintain the same wiring for this next step. You now need to input an impulse signal to the RC network under investigation. This is easily achieved by varying the duty cycle of the PULSE GENERATOR block as follows. Leave the FREQUENCY = 50 Hz, and set DUTY CYCLE to 0.05 (5%).

16. Notice how the output amplitude is diminished, and the rise time is still finite and easily visible. The input pulse is not a close enough approximation to a true impulse for our measurements. What is the pulse width currently?
17. Further reduce the DUTY CYCLE to 0.01 (1%) and increase the FREQUENCY to 100 Hz for viewing convenience, and to narrow the pulse further.

**Question 9**
What is the width if the impulse. What is its maximum amplitude?

18. Remove the scope lead to the input signal and view only the output impulse response. Trigger the scope and set the trig level to suit the signal size ie: it is now much less than 1 V.

19. Sketch the impulse response on the graph above. Use a new voltage scale for convenience.

From the preliminary discussion at the beginning of this lab, the unit impulse response of an RC network is:

\[ h(t) = \frac{1}{RC}e^{-t/RC}u(t) = 1000e^{-1000t}u(t) \]

**Question 10**
What is the equation for the measured impulse response using actual circuit values? How does this compare with theory?

**Question 11**
Explain why the impulse response reaches the peak value that it does.
HINT: superposition of 2 step responses is involved.
Response of the RC network to an exponential pulse

In this segment we investigate the response to a more general input. As an example we consider an exponential pulse with a different time constant. This provides an opportunity to revisit convolution as per Prep Question P3.

20. Switch the input signal for this next step to the ANALOG OUTPUT DAC-0. There is present an exponential pulse signal with the equation:

\[ x(t) = 1 e^{-500t} \cdot u(t) \]

![Figure 4: Wiring for RC network with exponential pulse input](image)

21. View both input and output signals. Trigger the scope and set the trig level to suit the signal size ie: it is now much less than 1 V.

22. Notice how the rise time of the output is easily visible with a particular time constant, and well as the decay time having its own time constant.

**Question 12**

What is the equation for the output signal and how does it compare with the theoretical output expected from this network? Refer to your work in preparation question 3.
23. Sketch the exponential pulse response, along with input, on the graph below.

Graph 2: exponential pulse response
Synthesising an RC NETWORK transfer function

24. In the preparation exercises it was shown that the impulse response of this RC NETWORK is given by:

\[ h(t) = \frac{1}{RC}.e^{-t/RC}.u(t) = 1000. e^{-1000t}.u(t) \]

and the corresponding system response in the Laplace domain by

\[ H(s) = \frac{1}{RC}/(s + (1/RC)) = \frac{1000}{s + 1000} \]

It was also shown that the Laplace domain equation can be modelled with the block diagram below,

where \( k/s \) represents integration with scaling factor \( k \).

Note that the value of \( k \) is set by the INTEGRATION RATE DIP switches.

Figure 5: block diagram of synthesised “RC network”

**Question 13**

Show that \( RC = |1/(k.a1)| \); where \( |k.a1| = 1000 \)

Figure 6: patching diagram of synthesised “RC network”
25. Patch together this system and investigate its performance using step and impulse responses as per settings below. Sketch the responses below.

Settings as as follows for step response:

FUNCTION GENERATOR: Select Squarewave output, 1Vpp, 0.5V offset, 100Hz frequency
SCOPE: Timebase 10ms
ADDER GAINS: a0 = 1.0; a1 = -0.1; a2 = 0
INTEGRATION RATE: DIPS set to UP:UP

For impulse response use the PULSE GENERATOR as a source with frequency = 100Hz, Duty cycle = 0.02 (2%)

Graph 3: step and impulse response of synthesized system

26. You would expect that a system like this will need a "small" amount of feedback as the change to the original step is not great. However the initial value suggested (-0.1) is not accurate enough. It is a good starting point as the time constant is similar. However the signal amplitude is too large.

27. Change the values of a0 and a1 until you get a perfect match with the actual RC NETWORK. To do this, view both outputs of the RC NETWORK and the synthesised "RC NETWORK" at point Y together and then make adjustments accordingly.
HINT: for convenient manual adjustment, you can set the GAIN ADJUST knob on the SIGEx board to vary the a1 gain coefficient, and then hand adjust the level.

Figure 7: Step responses: real RC network & synthesised "RC network". Not fully aligned.

Question 14
What values of a0 and a1 have you found give your synthesised system a perfect match to the actual RC NETWORK?

28. In order to calculate the actual transfer function of this synthesised system you will need to also know the INTEGRATION RATE of the integrator used.

In a previous experiment this was measured and you should use the same procedure to remeasure this value. In brief, input a bipolar squarewave to the integrator, with frequency around 300Hz, and amplitude less than 2Vpp from the FUNCTION GENERATOR. The INTEGRATION RATE is equal to ramp voltage spread / ramp time / input voltage. This value is the k value from Figure 5 above.

Question 15
What is the signal at the input to the integrator? Is this expected? Explain:
**Question 16**
Using the measured values above, what is the actual transfer function for your synthesised network which matches the actual RC network? Show your working.

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**Question 17**
Explain any discrepancies you find between expected theory and measurements. What sources of error are responsible for these?

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29. Use a varying sinusoid signal to plot the frequency response of the synthesised system and the actual RC network simultaneously. They should track each other.

Settings are as follows:
FUNCTION GENERATOR: Select SINEWAVE output, 8Vpp, 0V offset, 50Hz frequency
SCOPE: Timebase 20ms. Set Y AUTOSCALE to OFF
Start from 50 Hz, and take 10 measurements up to 2 kHz.

![Graph 4](image-url)

**Graph 4**: frequency response and bode plot of synthesized system

**Question 18**
Express the 3dB frequency in radians/sec and compare with your answer in Question 11(2)

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30. Sketch the bode plot for the system on the same graph as the frequency response from the previous step. Show your working.
Tutorial questions

Q1
(Convolution by graphical method)
In this exercise we use a graphical method to evaluate the convolution of two exponential functions, i.e., the impulse response of the RC circuit and the input signal in Task 21. The convolution formula is given in prep question P3 (a).

Note that you will need to carry out the multiplication of the two functions in the integrand for up to ten values of \( t \). This multiplication and plotting of the result as a function of the integration variable \( \tau \) could be done with the aid of a computer to save time. It will be evident that the area under the product will be small when \( t \) is large and also when near zero.

Compare the outcome with the theoretical result using the method in prep question P3 (a), and with your experimental records.

Q2
(re convolution theorem
(a) Use the results in Q. P5 to obtain the Laplace transforms of the input and the impulse response.

(b) Express the product of the transforms in (a) as a partial fraction sum.

(c) Obtain the inverse transform of the result in (b) and compare the outcome with the graph of the output response in T24.
N.B. There is no need to invoke the inversion formula - each of the two components in the partial fraction sum in (b) is an elementary transform that inverts by inspection).

(c) State the convolution theorem and the class of systems for which it applies. Does your result support the convolution theorem?

Q3
(re exponential input)
This question considers the possibility of using a calibrated exponential input to measure the time constant of the RC circuit. Suppose \( 1/\alpha_1 \) is the unknown time constant of the system under test and \( 1/\alpha_2 \) the time constant of the input exponential. Investigate the behaviour of the output as \( \alpha_2 \) is varied, focusing on the magnitude and position of the peak.