Algorithm Analysis

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More Examples

- Prove $n^2 + 3n + \log n$ is in $O(n^2)$
- Need to find $c$ and $n_0$ such that $n^2 + 3n + \log n \leq cn^2$ for $n \geq n_0$
- Proof:
  
  $n^2 + 3n + \log n \leq n^2 + 3n^2 + n$ for $n \geq 1$
  $\leq n^2 + 3n^2 + n^2$ for $n \geq 1$
  $\leq 5n^2$ for $n \geq 1$

  Therefore by definition $n^2 + 3n + \log n \in O(n^2)$.

  (Alternatively: $n^2 + 3n + \log n \leq n^2 + n^2 + n^2$ for $n \geq 10$
  $\leq 3n^2$ for $n \geq 10$)
More Examples

- Prove $n^2 + 3n + \log n$ is in $\Omega(n^2)$
- Want to find $c$ and $n_0$ such that
  \[ n^2 + 3n + \log n \geq cn^2 \text{ for } n \geq n_0 \]

\[ n^2 + 3n + \log n \geq n^2 \text{ for } n \geq 1 \]

$n^2 + 3n + \log n = O(n^2)$ and $n^2 + 3n + \log n = \Omega(n^2)$

$\Rightarrow n^2 + 3n + \log n = \Theta(n^2)$
The definitions imply a constant \( n_0 \) beyond which they are satisfied. We do not care about small values of \( n \).
Using limits to compare orders of growth

\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & c < 0 \\ \infty & c > 0 \end{cases} \]

- \( f(n) \in o(g(n)) \)
- \( f(n) \in O(g(n)) \)
- \( f(n) \in \Theta(g(n)) \)
- \( f(n) \in \Omega(g(n)) \)
- \( f(n) \in \omega(g(n)) \)
logarithms

- compare \( \log_2 n \) and \( \log_{10} n \)

- \( \log_a b = \log_c b / \log_c a \)

- \( \log_2 n = \log_{10} n / \log_{10} 2 \sim 3.3 \log_{10} n \)

- Therefore \( \lim(\log_2 n / \log_{10} n) = 3.3 \)

- \( \log_2 n = \Theta (\log_{10} n) \)
- Compare $2^n$ and $3^n$

- $\lim_{n \to \infty} \frac{2^n}{3^n} = \lim_{n \to \infty} (2/3)^n = 0$

- Therefore, $2^n \in o(3^n)$, and $3^n \in \omega(2^n)$

- How about $2^n$ and $2^{n+1}$?

  $2^n / 2^{n+1} = \frac{1}{2}$, therefore $2^n = \Theta(2^{n+1})$
L’ Hopital’s rule

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f(n)'}{g(n)'}
\]

Condition:
If both \( \lim f(n) \) and \( \lim g(n) \) are \( \infty \) or 0

\( \Rightarrow \) You can apply this transformation as many times as you want, as long as the condition holds.
Compare \( n^{0.5} \) and \( \log n \)

\[ \lim_{n \to \infty} \frac{n^{0.5}}{\log n} = ? \]

\( (n^{0.5})' = 0.5 n^{-0.5} \)

\( (\log n)' = 1 / n \)

\[ \lim (n^{-0.5} / 1/n) = \lim (n^{0.5}) = \infty \]

Therefore, \( \log n \in o(n^{0.5}) \)

In fact, \( \log n \in o(n^\varepsilon) \), for any \( \varepsilon > 0 \)
Stirling’s formula

\[ n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n = \sqrt{2\pi n^{n+1/2}} e^{-n} \]

\[ n! \approx \text{(constant)} \ n^{n+1/2} e^{-n} \]
Compare $2^n$ and $n!$

Therefore, $2^n = o(n!)$

Compare $n^n$ and $n!$

Therefore, $n^n = \omega(n!)$

How about $\log(n!)$?

\[
\lim_{n \to \infty} \frac{n!}{2^n} = \lim_{n \to \infty} \frac{c\sqrt{n}n^n}{2^n e^n} = \lim_{n \to \infty} c\sqrt{n} \left( \frac{n}{2e} \right)^n = \infty
\]

\[
\lim_{n \to \infty} \frac{n!}{n^n} = \lim_{n \to \infty} \frac{c\sqrt{n}n^n}{n^n e^n} = \lim_{n \to \infty} \frac{c\sqrt{n}}{e^n} = 0
\]
\[ \log(n!) = \log \frac{c \sqrt{n n^n}}{e^n} = C + \log n^{n+1/2} - \log(e^n) \]

\[ = C + n \log n + \frac{1}{2} \log n - n \]

\[ = C + \frac{n}{2} \log n + \left( \frac{n}{2} \log n - n \right) + \frac{1}{2} \log n \]

\[ = \Theta(n \log n) \]
More advanced dominance ranking

\[ \begin{align*}
    n! & \gg c^n \gg n^3 \gg n^2 \gg n^{1+\epsilon} \gg n \log n \gg n \gg \sqrt{n} \gg \\
    \log^2 n & \gg \log n \gg \log n / \log \log n \gg \log \log n \gg \alpha(n) \gg 1
\end{align*} \]
Asymptotic notations

- $O$: Big-Oh
- $\Omega$: Big-Omega
- $\Theta$: Theta
- $o$: Small-oh
- $\omega$: Small-omega

Intuitively:

$O$ is like $\leq$  
$o$ is like $<$  
$\Omega$ is like $\geq$  
$\omega$ is like $>$  
$\Theta$ is like $=$