Mathematical Review

Functions
Functions
Functions

- **Definition**: Let $A$ and $B$ be nonempty sets.
  - A function $f$ from $A$ to $B$, denoted $f: A \rightarrow B$ is an assignment of each element of $A$ to exactly one element of $B$.

- We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a$ of $A$.

- Functions are sometimes called **mappings** or **transformations**.
Functions

Given a function $f: A \rightarrow B$:

- We say $f$ maps $A$ to $B$ or $f$ is a mapping from $A$ to $B$.
- $A$ is called the domain of $f$.
- $B$ is called the codomain of $f$.
- If $f(a) = b$,
  - then $b$ is called the image of $a$ under $f$.
  - $a$ is called the preimage of $b$. 
Functions

- The **range** of $f$ is the set of all images of points in $A$ under $f$.
  - We denote it by $f(A)$.

- Two functions are equal when
  1. they have the same **domain**
  2. the same **codomain**
  3. map each element of the domain to the **same element** of the codomain
Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment. 
    Students and grades example.
  - A formula.
    \( f(x) = x + 1 \)
  - A computer program.
    - A Java program that when given an integer \( n \), produces the \( n \)th Fibonacci Number
Questions

\( f(a) = ? \)

The image of \( d \) is ? \( z \)

The domain of \( f \) is ? \( A \)

The codomain of \( f \) is ? \( B \)

The preimage of \( y \) is ? \( b \)

\( f(A) = ? \) \( \{y,z\} \)

The preimage(s) of \( z \) is (are) ? \( \{a,c,d\} \)
Question on Functions and Sets

- If \( f : A \rightarrow B \), then

\[
f\{a,b,c,d\} \text{ is } \{y,z\} \\
f\{c,d\} \text{ is } \{z\}
\]
**Definition**: A function $f$ is said to be **one-to-one**, or **injective**, if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in the domain of $f$.

- A function is said to be an **injection** if it is a one-to-one mapping.
Surjections

- **Definition**: A function \( f \) from \( A \) to \( B \) is called **onto** or **surjective**, if and only if for every element \( b \in B \) there is an element \( a \in A \) with \( f(a) = b \).

- A function \( f \) is called a **surjection** if it is **onto**.

![Diagram of surjections with elements a, b, c, d in A and x, y, z in B, showing surjective mappings.]
Bijections

**Definition:** A function $f$ is a *one-to-one correspondence*, or a *bijection*, if it is *both* one-to-one and onto (surjective and injective).
Examples of Different Correspondences

(a) One-to-one, not onto

(b) Onto, not one-to-one

(c) One-to-one, and onto

(d) Neither one-to-one nor onto

(e) Not a function
Showing that $f$ is one-to-one or onto

- **Example 1**: Let $f$ be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$.
  - Is $f$ an onto function?

- **Solution**: Yes, $f$ is onto since all three elements of the codomain are images of elements in the domain.
  - If the codomain were changed to $\{1,2,3,4\}$, $f$ would not be onto.

- **Example 2**: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

- **Solution**: No, $f$ is not onto because there is no integer $x$ with $x^2 = -1$ (negative integers), for example.
Inverse Functions

- **Definition:** Let $f$ be a bijection from $A$ to $B$.
  - Then the *inverse* of $f$, denoted $f^{-1}$, is the function from $B$ to $A$ defined as
    $$f^{-1}(y) = x \iff f(x) = y$$
  - No inverse exists unless $f$ is a bijection.
Inverse Functions

\[ f \] \hspace{2cm} \[ f^{-1} \]

\begin{align*}
A & \xrightarrow{f} B \\
\text{a} & \rightarrow \text{v} \\
\text{b} & \rightarrow \text{w} \\
\text{c} & \rightarrow \text{x} \\
\text{d} & \rightarrow \text{y}
\end{align*}

\begin{align*}
A & \xleftarrow{f^{-1}} B \\
\text{a} & \leftarrow \text{v} \\
\text{b} & \leftarrow \text{w} \\
\text{c} & \leftarrow \text{x} \\
\text{d} & \leftarrow \text{y}
\end{align*}
Example 1: Let $f$ be the function from $\{a, b, c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.

- Is $f$ invertible and if so what is its inverse?

Solution: The function $f$ is invertible because it is a one-to-one and onto correspondence.

The inverse function $f^{-1}$ reverses the correspondence given by $f$, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$. 
Questions

- **Example 2**: Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be such that $f(x) = x + 1$.
  - Is $f$ invertible, and if so, what is its inverse?

**Solution**: The function $f$ is invertible because it is a one-to-one and onto correspondence.

The inverse function $f^{-1}$ reverses the correspondence so $f^{-1}(y) = y - 1$. 
Questions

• Example 3: Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^2$.
  • Is $f$ invertible, and if so, what is its inverse?

Solution: The function $f$ is not invertible because it is not one-to-one.
Composition

- **Definition**: Let two functions \( f \): \( B \rightarrow C \), \( g \): \( A \rightarrow B \).
- The **composition** of \( f \) with \( g \), denoted \( f \circ g \) is the function from \( A \) to \( C \) defined by
  \[
  f \circ g(x) = f(g(x))
  \]
Composition

\[ A \xrightarrow{g} B \xrightarrow{f} C \]

\[ A \xrightarrow{f \circ g} C \]
Composition

Example 1: If \( f(x) = x^2 \) and \( g(x) = 2x + 1 \), then

\[
f(g(x)) = (2x + 1)^2
\]

and

\[
g(f(x)) = 2x^2 + 1
\]
Graphs of Functions

- Let $f$ be a function from the set $A$ to the set $B$.
- The **graph** of the function $f$ is the set of ordered pairs \{$(a,b) \mid a \in A$ and $f(a) = b$\}. 

Graph of $f(n) = 2n + 1$ from $\mathbb{Z}$ to $\mathbb{Z}$  

Graph of $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$