CMPS 3120

Algorithm Analysis

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Decrease-by-Constant-Factor Algorithms

In this variation of decrease-and-conquer, instance size is reduced by the same factor (typically, 2)

Examples:
- binary search and the method of bisection
- exponentiation by squaring
- multiplication à la russe (Russian peasant method)
- fake-coin puzzle
- Josephus problem
Binary Search

Very efficient algorithm for searching in sorted array:

\[ K \]

vs

\[ A[0] \ldots A[m] \ldots A[n-1] \]

If \( K = A[m] \), stop (successful search); otherwise, continue searching by the same method in \( A[0..m-1] \) if \( K < A[m] \) and in \( A[m+1..n-1] \) if \( K > A[m] \)

\( l \leftarrow 0; \ r \leftarrow n-1 \)

while \( l \leq r \) do

\( m \leftarrow \lfloor (l+r)/2 \rfloor \)

if \( K = A[m] \) return \( m \)

else if \( K < A[m] \) \( r \leftarrow m-1 \)

else \( l \leftarrow m+1 \)

return -1
Analysis of Binary Search

- Time efficiency
  - worst-case recurrence: $C_w(n) = 1 + C_w(\lfloor n/2 \rfloor)$, $C_w(1) = 1$
  - solution: $C_w(n) = \lceil \log_2(n+1) \rceil$

  This is VERY fast: e.g., $C_w(10^6) = 20$

- Optimal for searching a sorted array
- Limitations: must be a sorted array (not linked list)
- Bad (degenerate) example of divide-and-conquer
- Has a continuous counterpart called bisection method for solving equations in one unknown $f(x) = 0$ (see Sec. 12.4)
Exponentiation by Squaring

The problem: Compute $a^n$ where $n$ is a nonnegative integer

The problem can be solved by applying recursively the formulas:

**For even values of $n$**

$$a^n = (a^{n/2})^2 \text{ if } n > 0 \text{ and } a^0 = 1$$

**For odd values of $n$**

$$a^n = (a^{(n-1)/2})^2 a$$

Recurrence: $M(n) = M(\lfloor n/2 \rfloor) + f(n)$, where $f(n) = 1$ or $2$,

$M(0) = 0$

**Master Theorem:** $M(n) \in \Theta(\log n) = \Theta(b)$ where $b = \lceil \log_2(n+1) \rceil$

Russian Peasant Multiplication

The problem: Compute the product of two positive integers

Can be solved by a decrease-by-half algorithm based on the following formulas.

For even values of \( n \):

\[ n * m = \frac{n}{2} * 2m \]

For odd values of \( n \):

\[ n * m = \frac{n-1}{2} 2m + m \text{ if } n > 1 \text{ and } m \text{ if } n = 1 \]
Example of Russian Peasant Multiplication

Compute $20 \times 26$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>26</td>
</tr>
<tr>
<td>10</td>
<td>52</td>
</tr>
<tr>
<td>5</td>
<td>104</td>
</tr>
<tr>
<td>2</td>
<td>208</td>
</tr>
<tr>
<td>1</td>
<td>416</td>
</tr>
</tbody>
</table>

$520$

Note: Method reduces to adding $m$'s values corresponding to odd $n$'s.
Fake-Coin Puzzle (simpler version)

There are $n$ identically looking coins one of which is fake. There is a balance scale but there are no weights; the scale can tell whether two sets of coins weigh the same and, if not, which of the two sets is heavier (but not by how much). Design an efficient algorithm for detecting the fake coin. Assume that the fake coin is known to be lighter than the genuine ones.

Decrease by factor 2 algorithm

Decrease by factor 3 algorithm