

- (1) At each stage of the Euclidean Algorithm running on two integers a, b , it is possible to express the remainder r as a linear combination of a and b . For example, let $a = 3094$ and $b = 2513$, then successive steps of the Euclidean Algorithm give:

$$3094 = 1 \cdot 2513 + \mathbf{581}$$

$$2513 = 4 \cdot 581 + \mathbf{189}$$

$$581 = 3 \cdot 189 + \mathbf{14}$$

$$189 = 13 \cdot 14 + \mathbf{7}$$

$$14 = 2 \cdot 7 + \mathbf{0}$$

Also notice that

$$3094 \cdot 1 + 2513 \cdot (-1) = \mathbf{581}$$

$$3094 \cdot (-4) + 2513 \cdot 5 = \mathbf{189}$$

$$3094 \cdot 13 + 2513 \cdot (-16) = \mathbf{14}$$

$$3094 \cdot (-173) + 2513 \cdot (213) = \mathbf{7}$$

$$3094 \cdot 359 + 2513 \cdot (-442) = \mathbf{0}$$

- (a) Find the linear combination of the pairs of numbers in the following that gives their respective gcd. Say, in the example, 7 is the gcd of 2513 and 3094, and we can get $2513 \cdot 213 + 3094 \cdot (-173) = 7$ (Hint: try working backwards from the Euclidean Algorithm steps.)

(a) 484 and 451

(c) 3953 and 1829

(b) 5280 and 3600

(d) 144 and 89

- (e) Is it possible to build up these numbers while running the Euclidean Algorithm? That is, say, can I find a linear combination of 3094 and 2513 to get 189 without having to evaluate “backwards”? (Hint: use the numeric examples that you have developed as your basis of building up the algorithm.)

Definition. An equation of the form $ax + by = c$ where a, b, c, x, y are all integers is called a Diophantine equation.

- (2) Find integer solutions for the following:

(a) $21x + 35y = 7$

(c) $143x + 253y = 165$

(b) $11x + 15y = 1$

(d) $14x + 18y = 5$

- (3) From the above, can you make a conjecture on when a Diophantine equation has a solution? (Can you argue why this is true?)
- (4) For each of the above Diophantine equations with a solution, find at least one more distinct solution.
- (5) How many solutions are there in each of the above Diophantine equations? Can you find all of them? Find a *formula* to generate lots of solutions.
- (6) A trucking company has to move 844 refrigerators. It has two types of trucks it can use; one carries 28 refrigerators and the other 34 refrigerators. If it only sends out full trucks and all the trucks return empty, list the possible ways of moving all the refrigerators.
- (7) Show how to to measure exactly 2 liters of water from a river using a 27 liter jug and a 16 liter jug. If you could not lift the larger jug when full but could push it over, could you still measure the 2 liters?