Reduction of Discontinuity for Derivations on Fréchet Algebras

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Abstract. We consider with which strictly irreducible representations the discontinuity of a derivation on a (locally multiplicatively convex) Fréchet algebra must be associated. We consider only those strictly irreducible representations which are compatible with the topology of the algebra (see Definition 1.3). Our main results are Proposition 2.7 and Proposition 3.5, which together show that when consideration is fixed upon each seminorm (see Definition 3.2), the exceptional set of primitive ideals supporting the discontinuity must be a finite set, with each ideal being the kernel of some finite dimensional irreducible representation. This result is best possible since Charles Read has constructed a radical Fréchet algebra with identity adjoined which has a derivation whose separating ideal is the entire algebra and one could take (countable) Fréchet products of his counterexample. We also prove that derivations on commutative Fréchet algebras whose structure spaces are compact metric in the weak* topology have only finitely many such exceptional points overall.

§1. Introduction.

In our previous paper [Thomas4] we demonstrated a strong connection between formal power series quotients of commutative Fréchet algebras and (possibly discontinuous) derivations. In the sufficient conditions [Thomas4, Theorem 3.10] we focused attention on radical Fréchet algebras with identity adjoined because we were not able to give a reduction of the problem of where the discontinuity of a derivation on a Fréchet algebra is localized in terms of primitive ideals. In this paper we consider derivations on both commutative and non-commutative Fréchet algebras and we give such a reduction, although we have to assume that the strictly irreducible representations under consideration are compatible with the topology of the algebra (see Definition 1.3).

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Throughout the paper $\mathcal{A}$ will denote a (locally multiplicatively convex) Fréchet algebra over the complex field. It is sometimes convenient to have an identity element $1$. If $\mathcal{A}$ already has an identity element we will let $\mathcal{A}^d$ denote $\mathcal{A}$. Otherwise we will adjoin an identity in the usual way so that $\mathcal{A}^d \cong \mathbb{C} 1 \oplus \mathcal{A}$. We will assume without loss of generality that the locally convex topology $\tau$ on $\mathcal{A}$ arises from a countable increasing family of submultiplicative seminorms $\{ \| \cdot \|_n \mid n \in \mathbb{N} \}$. This means that the following conditions are satisfied.

(1.1a) for all $a \in \mathcal{A}$ $\|a\|_n \leq \|a\|_m$ whenever $n \leq m$ in $\mathbb{N}$, and $\|a\|_n = 0$ for all $n \in \mathbb{N}$ if and only if $a = 0$.

(1.1b) $\|\lambda a\|_n = |\lambda| \|a\|_n$ for all $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, and $n \in \mathbb{N}$.

(1.1c) $\|a + b\|_n \leq \|a\|_n + \|b\|_n$ for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$.

(1.1d) $\|ab\|_n \leq \|a\|_n \|b\|_n$ for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$.

(1.1e) $\lim_{i \to \infty} a_i = a$ in the Fréchet topology $\tau$ of $\mathcal{A}$ if and only if for each fixed $n \in \mathbb{N}$ we have that $\lim_{i \to \infty} \|a - a_i\|_n = 0$.

If $\mathcal{A}$ does not have an identity element we extend each of these seminorms to $\mathcal{A}^d$ in the usual way, that is

$$\|\lambda 1 + a\|_n = |\lambda| + \|a\|_n$$

for $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$. It is routine to check that (1.1a–e) still hold. In the very special case that the Fréchet algebra $\mathcal{A}$ is actually a Banach algebra then for some $n \in \mathbb{N}$ the seminorm $\| \cdot \|_n$ is a norm and for $m \geq n$ each $\| \cdot \|_m$ depends continuously on $\| \cdot \|_n$.

**Definition 1.2.**  A derivation on a (possibly non-commutative) Fréchet algebra $\mathcal{A}$ is a linear map $D$ from $\mathcal{A}$ to itself satisfying

$$D(ab) = a(Db) + (Da)b$$

for all $a, b$ in $\mathcal{A}$. We emphasize that we do not require $D$ to be continuous.

Note that a derivation $D$ extends to $\mathcal{A}^d$ by (if necessary) defining $D1 = 0$.

Throughout this paper we use the terms subalgebra, ideal, and subspace, in the algebraic sense, that is, we do not assume these substructures are closed.

Much the same general strategy used to analyze a (possibly discontinuous) derivation on a Banach algebra (see reductions of of B. E. Johnson [Johnson] and
Thomas [Thomas2]) can be used to analyze a derivation on a locally multiplicatively convex Fréchet algebra.

In the case of a Banach algebra continuity depends upon the invariance of primitive ideals (see Lemma 1.4). The work of Johnson and Sinclair (see [Johnson] and [Johnson-Sinclair]) showed that the discontinuity of a derivation on a Banach algebra is localized on finitely many primitive ideals each of finite codimension. From this reduction, one can prove that if there exists a discontinuous derivation on a (non-commutative, see [Thomas2][Thomas3]) Banach algebra and the derivation fails to leave some primitive ideal invariant then there exists a non-commutative radical Banach algebra with a local power series quotient. If it could then be shown that no such radical Banach algebra exists, then every derivation on a Banach algebra would leave the primitive ideals invariant, an assertion known as the non-commutative Singer-Wermer conjecture. It is currently unproven.

In the more general setting of Fréchet algebras continuity does not generally imply invariance and one can even find continuous derivations which do not leave the radical invariant, for example, formal differentiation on the Fréchet algebra of formal power series over the complex field \( \mathcal{C}[[X]] \). However, the question of where the discontinuity is localized is still quite important. Allowances must be made for the possible existence of locally nilpotent, non-nilpotent elements (see [Thomas4, Theorem 3.10]).

Very importantly, Charles Read has shown that there exists a radical Fréchet algebra with identity adjoined which has a discontinuous derivation whose separating ideal \( S(D) \) is equal to the entire algebra [Read]. It is easy to see that one could take a (countable) Fréchet product of this example so that for each fixed seminorm, there would be finitely many primitive ideals of finite codimension supporting discontinuity of the derivation (with a countable number of primitive ideals in all). Our main goal is to prove that this is as extensive as the discontinuity can be. Proposition 2.7 shows that infinite dimensional representations cannot support discontinuity and Proposition 3.5 covers the case of finite dimensional representations.

We will start our reduction using a combination of representation theory together with the theory of automatic continuity. When one considers the strictly irreducible representations of a Fréchet algebra \( \mathcal{A} \) some complications occur. Even when \( \mathcal{A} \) is commutative, in which case the finite dimensional representations are 1-dimensional, corresponding to characters \( \varphi \) from \( \mathcal{A} \) into the complex field, it is
not immediate that a character $\varphi$ must be continuous and the kernel of $\varphi$ (which is a maximal ideal) therefore closed. This is the classical Michael’s problem (see [Michael] and, for a reduction of the problem, [Esterle]). It is currently unproven.

We will have to make certain assumptions which we collect in Definition 1.3. We will observe the customary notational conventions. Let $\text{irred}(\mathcal{A})$ be the full set of non-isomorphic strictly irreducible representations of $\mathcal{A}$ as linear transformations (elements of $\mathcal{L}(X)$) on some complex vector space $X$. Let $\text{dim}(\pi)$ be the dimension of $X$ over the complex field, and for $n \in \mathbb{N}$, let $\mathcal{M}_n$ be the algebra of $n$ by $n$ matrices with entries in the complex field $\mathbf{C}$. So if $\text{dim}(\pi) = n \in \mathbb{N}$ then $\pi$ is a homomorphism from $\mathcal{A}$ into $\mathcal{M}_n$. An ideal $P$ is primitive if it is the kernel of some strictly irreducible representation $\pi$ and we write $P = \text{kernel}(\pi)$. Let $\text{prim}(\mathcal{A})$ be the set of all primitive ideals of $\mathcal{A}$.

**Definition 1.3.** Let $\mathcal{A}$ be a Fréchet algebra over the complex field. We define the subset $\text{Irred}(\mathcal{A}) \subseteq \text{irred}(\mathcal{A})$ to consist of those $\pi \in \text{irred}(\mathcal{A})$ which are compatible with the topology of $\mathcal{A}$ in the following sense:

1. (1.3a) If $X$ is the complex vector space on which $\pi(\mathcal{A})$ acts we require that for each $\zeta \in X$ the annihilator
   \[ \text{ann}(\zeta) = \{ a \in \mathcal{A} \mid \pi(a)\zeta = 0 \} \text{ is closed in } \mathcal{A} \]

We define the subset $\text{Prim}(\mathcal{A}) \subseteq \text{prim}(\mathcal{A})$ to consist of those primitive ideals $P \in \text{prim}(\mathcal{A})$ such that $P$ is the kernel of some $\pi \in \text{Irred}(\mathcal{A})$.

We note that for a Banach algebra $\mathcal{A}$ it is a routine application of the fact that the group of units is open to show that maximal modular left ideals are closed (see [Rickart] or [Bonsall-Duncan]). From this it follows naturally that $\text{irred}(\mathcal{A}) = \text{Irred}(\mathcal{A})$ and $\text{prim}(\mathcal{A}) = \text{Prim}(\mathcal{A})$. If $\mathcal{A}$ is a Fréchet algebra and $\pi \in \text{Irred}(\mathcal{A})$ is a finite dimensional representation such that all annihilators are closed then it is easy to see that the Banach algebra argument can be followed and that an equivalent statement of (1.3a) would be

1. (1.3b) (special case of $\text{dim}(\pi) < \infty$) If $X$ is the complex vector space on which $\pi(\mathcal{A})$ acts then $X$ can be given a Banach space topology for which $\pi(a)$ is a bounded linear operator for each $a \in \mathcal{A}$ and for which the map $a \rightarrow \pi(a)$
is continuous.
In the case of a Fréchet algebra which is not a Banach algebra we are essentially
narrowing our focus to those strictly irreducible representations which are compat-
ible with the topology of the algebra. We shall generally say irreducible for strictly
irreducible in the remainder of the paper.

Turning to the theory of automatic continuity for linear functions on Fréchet
spaces, a general reference is [Thomas1]. The automatic continuity results for
Fréchet spaces correspond nicely to the results for Banach spaces (see [Sinclair])
except that closures have to be considered with respect to each seminorm.

In both of the following sections we will need the well known result from the
theory of automatic continuity:

**Lemma 1.4.** Let $D$ be a (possibly discontinuous) derivation on a Fréchet
algebra $\mathcal{A}$. Let $I$ be a closed ideal of $\mathcal{A}$ and let $Q_I$ denote the continuous canonical
quotient map from $\mathcal{A}$ onto $\mathcal{A}/I$. For each $k \in \mathbb{N}$ define the separating subspace of
$D^k$ as follows

\[ S(D^k) \equiv \{ z \in \mathcal{A} \mid \exists x_i \to 0 \text{ in } \mathcal{A} \text{ with } D^k(x_i) \to z \} \]

The following conditions are equivalent:

(i.) $Q_I D^k$ is continuous for all $k \in \mathbb{N}$.

(ii.) $S(D^k) \subseteq I$ for all $k \in \mathbb{N}$.

If the codimension $[\mathcal{A} : I]$ is finite then the following is also equivalent to the above
two conditions.

(iii.) $Q_I D^k$ is continuous when restricted to the closed ideal $I$ for all $k \in \mathbb{N}$.

In the special case where $\mathcal{A}$ is actually a Banach algebra and $I \subseteq \text{prim}(\mathcal{A})$, either
(i.) or (ii.) is equivalent to the following invariance condition:

(iv.) $D(I) \subseteq I$.

**Proof.** Assume that condition (i.) holds. It is fundamental (see [Thomas1,
discussion on results (1)–(3), pages 518–519]) that $Q_I D^k$ will be continuous on $\mathcal{A}$
precisely when $Q_I(S(D^k)) = \{0\}$. This will happen precisely when $S(D^k) \subseteq I$.
Hence, condition (i.) is equivalent to condition (ii.).

Suppose the codimension $[\mathcal{A} : I]$ is finite and assume that condition (iii.) holds.
Since a finite extension of a continuous linear mapping on a closed subspace is
continuous it is clear the condition (i.) must hold also. Clearly condition (i.)
implies condition (iii.). Hence they are equivalent.

Suppose that $\mathcal{A}$ is a Banach algebra, $I \in \text{prim}(\mathcal{A})$, and condition (i.) above holds. An application of [Thomas3, Lemma 1.1] shows that there exists a constant $C > 0$ such that

$$
\|Q_I D^k\| \leq C^k
$$

for all $k \in \mathbb{N}$ (it is not necessary for $I$ to be primitive for this). Now, given that $I$ is primitive, apply [Thomas3, Lemma 1.2] and this shows that $D(I) \subseteq I$. Conversely, suppose that $D(I) \subseteq I$. Then $D$ drops to a derivation $\tilde{D}$ on the semisimple quotient Banach algebra $\mathcal{A}/I$. Applying [Johnson-Sinclair, Theorem 4.1] we see that $\tilde{D}$ is continuous, and hence $\tilde{D}^k$ is continuous for all $k \in \mathbb{N}$. The invariance of $I$ under $D$ shows that $Q_I D^k$ factors through $\mathcal{A}/I$ and that $Q_I D^k = (\tilde{D}^k)Q_I$ which is continuous for all $k \in \mathbb{N}$, showing that condition (i.) holds.

The classical uses of derivatives on function spaces depend upon derivatives not leaving primitive ideals invariant, but continuity is always desirable. With this in mind we make the following definition.

**Definition 1.5.** Let $D$ be a (possibly discontinuous) derivation on a Fréchet algebra $\mathcal{A}$. Let $\pi \in \text{Irred}(\mathcal{A})$ and let $P = \text{kernel}(\pi) \in \text{Prim}(\mathcal{A})$.

i. If $D(P) \subseteq P$ we say that $D$ is invariant at $P$.

ii. If for each $N \in \mathbb{N}$ there exists $C > 0$ and $M \in \mathbb{N}$ such that

$$
\|Q_P D^k a\|_N \leq C^k(k!)\|a\|_M
$$

for every $a \in \mathcal{A}$ we say that $D$ is analytic at $P$.

iii. If $D$ is not analytic at $P$ but $Q_P D^k$ is continuous for every $k \in \mathbb{N}$ we say that $D$ is continuously differentiable at $P$.

iv. If $Q_P D^k$ is discontinuous for some $k \in \mathbb{N}$ we say that $D$ is singular at $P$ and we say that either the primitive ideal $P$ or the strictly irreducible representation $\pi$ supports the discontinuity of $D$.

Note that case (i.) can, in general, overlap with the other cases, although cases (ii.), (iii.), and (iv.) are distinct. The unproven non-commutative Singer-Werner conjecture states that if $\mathcal{A}$ is a Banach algebra then case (i.) always holds.
In this paper we are, of course, primarily interested in the singular case of Definition 1.5. The techniques for dealing with $\pi \in \text{Irred}(\mathcal{A})$ differ somewhat depending upon whether $\dim(\pi)$ is infinite or finite. Consequently we handle these two cases in separate sections.
§2. Infinite dimensional irreducible representations of Fréchet Algebras.

Let $D$ be a derivation on a Fréchet algebra $\mathcal{A}$. Let $\pi \in \text{Irred}(\mathcal{A})$ with $\dim(\pi) = \infty$ and let $P = \text{kernel}(\pi)$. Our goal is to show that $Q_P D^k$ is continuous for all $k \in \mathbb{N}$ so that $\pi$ cannot support discontinuity of $D$. In consequence, any discontinuity of $D$ must be localized in the finite dimensional irreducible representations of $\mathcal{A}$.

Before developing the lemmas needed for this result it might be best to consider an example of such an infinite dimensional representation.

**Example 2.1.** Let $A(\Delta)$ be the disk algebra of functions analytic on the open unit disk $\Delta$ and continuous on the closed unit disk. With the usual supremum norm, $\| \cdot \|$, $A(\Delta)$ is a commutative Banach algebra. Let $\mathcal{O}$ be the Fréchet algebra of functions $f$ analytic on all of $\mathcal{C}$ with the usual increasing family of seminorms

$$\|f\|_n = \sup\{|f(\zeta)| : |\zeta| \leq n\}$$

defined for each $n \in \mathbb{N}$. Note that these seminorms are actually (incomplete) norms. Clearly we can regard $\mathcal{O}$ as being embedded in $A(\Delta)$ as a dense subspace.

Let $B(A(\Delta))$ be the non-commutative Banach algebra of all continuous linear operators on $A(\Delta)$ (ignoring the multiplication and regarding $A(\Delta)$ as a Banach space only). Define

$$\mathcal{A}_0 \equiv \{T \in B(A(\Delta)) : T(A(\Delta)) \subseteq \mathcal{O} \text{ and } \text{rank}(T) < \infty\}$$

It is routine to show that $\mathcal{A}_0$ is an algebra over $\mathcal{C}$. It is also clear that $\mathcal{A}_0$ acts irreducibly on $\mathcal{O}$ (although not on $A(\Delta)$). Note that

$$\|Tf\|_1 = \|Tf\| \leq \|T\| \|f\| = \|T\|_1 \|f\|_1$$

for all $T \in \mathcal{A}_0$ and $f \in \mathcal{O}$. Let $q \in \mathbb{N}$ with $q \geq 2$. Let $T \in \mathcal{A}_0$. Since $T(\mathcal{O})$ is a finite dimensional vector space all norms on it are equivalent so there exists a constant $C_q$ such that

$$\|Tf\|_q \leq C_q\|T\|_1 \|f\| \leq C_q \|T\|_q \|f\| \leq (C_q \|T\|_1) \|f\|_q$$

for all $f \in \mathcal{O}$. Define seminorms on $\mathcal{A}_0$ as follows:

$$\|T\|_q \equiv \sup_{f \neq 0 \text{ in } \mathcal{O}} \frac{\|Tf\|_q}{\|f\|_q}$$
for \( q = 1, 2, 3, \ldots \). It is then routine to check that these seminorms are submultiplicative, that \( \| \cdot \|_1 \) is the operator norm inherited from \( \mathcal{B}(A(\Delta)) \), and that

\[
\| T f \|_q \leq \| T \|_q \| f \|_q
\]

for all \( q \in \mathbb{N} \), \( T \in \mathcal{A}_0 \) and \( f \in \mathcal{O} \). Let \( \mathcal{A} \) be the Fréchet algebra which is the completion of \( (\mathcal{A}_0, \| \cdot \|_q) \) with respect to any translation invariant metric given by this family of seminorms. It follows that the action of \( \mathcal{A}_0 \) on \( \mathcal{O} \) extends continuously to \( \mathcal{A} \) and that

\[
\| T f \|_q \leq \| T \|_q \| f \|_q
\]

for all \( q \in \mathbb{N} \), \( T \in \mathcal{A} \) and \( f \in \mathcal{O} \). Since \( \mathcal{A}_0 \) acts irreducibly on \( \mathcal{O} \) so does \( \mathcal{A} \), with \( \pi \) being the obvious embedding of \( \mathcal{A} \) into \( \mathcal{L}(\mathcal{O}) \). However, since \( \| \cdot \|_1 \) equals the operator norm in \( \mathcal{B}(A(\Delta)) \) we can also regard \( \mathcal{A} \) as being embedded into the closure of the finite rank operators (i.e. the compact operators) on \( A(\Delta) \). Hence, \( \ker(\pi) \) is \( \{0\} \) and \( \pi \) is a faithful irreducible representation of \( \mathcal{A} \).

We have the following lemma which shows that certain features of the above example are always present.

**Lemma 2.2.** Let \( \mathcal{A} \) be a Fréchet algebra. Let \( \pi \in \text{Irred}(\mathcal{A}) \) be an irreducible representation of \( \mathcal{A} \) on an infinite dimensional complex vector space \( X \). Let \( P = \ker(\pi) \). Then \( X \) can be given a Fréchet space topology with separating increasing family of seminorms \( \{ \| \cdot \|_n \mid n \in \mathbb{N} \} \) satisfying

\[
\| \pi(a) \zeta \|_q \leq \| a \|_q \| \zeta \|_q
\]

for all \( q \in \mathbb{N} \), \( a \in \mathcal{A} \), and \( \zeta \in X \). In addition, \( P \) is a closed ideal of \( \mathcal{A} \) and each seminorm on \( X \) is either trivial or a (possibly incomplete) norm.

**Proof.** Fix any non-zero vector \( \zeta_0 \in X \), choose \( e \in \mathcal{A} \) satisfying \( \pi(e) \zeta_0 = \zeta_0 \), and note that the annihilator and left ideal

\[
\mathcal{M} \equiv \{ c \in \mathcal{A} \mid \pi(c) \zeta_0 = 0 \}
\]

is closed and modular since \( a - ae \in \mathcal{M} \) for all \( a \in \mathcal{A} \). Hence \( e \) is a right modular unit for \( \mathcal{M} \). Jacobson density shows that \( \mathcal{A}/\mathcal{M} \) is \( \mathcal{A} \)-module isomorphic to \( X \) under the map

\[
\varphi : a + \mathcal{M} \to \pi(a) \zeta_0
\]
for $a \in \mathcal{A}$. Since $\mathcal{A}/\mathcal{M}$ has a Fréchet space topology inherited from $\mathcal{A}$ and given by seminorms
\[
\|a + \mathcal{M}\|_q \equiv \inf_{m \in \mathcal{M}} \|a + m\|_q
\]
for $q \in \mathbb{N}$ we can transfer this Fréchet space topology to $X$ via $\|\zeta\|_q \equiv \|\varphi^{-1}(\zeta)\|_q$ so that $\|\zeta\|_q = \|b + \mathcal{M}\|_q$ for any $b \in \mathcal{A}$ satisfying $\pi(b)\zeta_0 = \zeta$. Let $q \in \mathbb{N}$, $a \in \mathcal{A}$ and $\zeta \in X$. Let $b \in \mathcal{A}$ satisfy $\pi(b)\zeta_0 = \zeta$, and note that
\[
\|\pi(a)\zeta\|_q = \|\pi(ab)\zeta_0\|_q = \|ab + \mathcal{M}\|_q \leq \|a\|_q\|b + \mathcal{M}\|_q = \|a\|_q\|\zeta\|_q
\]
The fact that $P = \cap_{\zeta \in X} \text{ann}(\zeta)$ shows that $P$ is closed. Finally, suppose that $\|\zeta\|_q = 0$ for some non-zero vector $\zeta \in X$ and $q \in \mathbb{N}$. If $\|\cdot\|_q$ is not trivial (i.e. $\mathcal{M}$ is not $\|\cdot\|_q$-dense in $\mathcal{A}$) choose $\eta \in X$ with $\|\eta\|_q \neq 0$. Jacobson density shows that there exists $a \in \mathcal{A}$ with $\pi(a)\zeta = \eta$. Hence
\[
0 \neq \|\eta\|_q = \|\pi(a)\zeta\|_q \leq \|a\|_q\|\zeta\|_q
\]
a contradiction. This completes the proof of the lemma.

We will henceforth assume that for any such irreducible representation the vector space $X$ has been endowed with the topology ensured by Lemma 2.2.

**Lemma 2.3.** Let $\mathcal{A}$ be a Fréchet algebra. Let $\pi \in \text{Irred}(\mathcal{A})$ be an irreducible representation of $\mathcal{A}$ on an infinite dimensional complex vector space $X$. Let $P = \text{ker}(\pi)$. Let $D$ be a derivation on $\mathcal{A}$. Fix $k \in \mathbb{N}$. The map $QPD^k$ is continuous if and only if the map $x \to \pi(D^kx)\zeta$ is continuous for each fixed $\zeta \in X$.

**Proof.** Fix $k \in \mathbb{N}$ and assume that the map $QPD^k$ is continuous. By Lemma 1.4 it follows that $\mathcal{S}(D^k) \subseteq P$. For each $\zeta \in X$ define $\varphi_\zeta(y) = \pi(y)\zeta$. By Lemma 2.2 it follows that each map $\varphi_\zeta$ is continuous from $\mathcal{A}$ to $X$ and annihilates $P$. Therefore, for each $\zeta \in X$, $\varphi_\zeta(\mathcal{S}(D^k)) = \{0\}$, which shows that $\varphi_\zeta \circ D^k$ is continuous. But $(\varphi_\zeta \circ D^k)(x) = \pi(D^kx)\zeta$, which finishes the proof of the first half of the lemma.
Now assume that the map \( x \to \pi(D^k x)\zeta \) is continuous for each fixed \( \zeta \in X \). Define the continuous maps \( \varphi_\zeta \) as above and suppose that \( x \in \mathcal{S}(D^k) \), i.e. there exists \( x_i \to 0 \) in \( \mathcal{A} \) with \( D^k x_i \to x \). Since \( x_i \to 0 \) it follows that

\[
\pi(D^k x_i)\zeta \to 0
\]

Since \( D^k x_i \to x \) and since \( \varphi_\zeta \) is continuous it also follows that

\[
\pi(D^k x_i)\zeta = \varphi_\zeta(D^k x_i) \to \varphi_\zeta(x) = \pi(x)\zeta
\]

Therefore \( \pi(x)\zeta = 0 \). Since \( \zeta \) was arbitrary in \( X \) this shows \( x \in \cap_{\zeta \in X} \text{ann}(\zeta) = \ker(\pi) = P \). Therefore \( \mathcal{S}(D^k) \subseteq P \), or, equivalently, \( Q_PD^k \) is continuous, finishing the second half of the lemma.

We proceed with the following lemmas which are very much in the spirit of Johnson and Sinclair [Johnson-Sinclair, Lemma 2.1 and Lemma 2.2] with modifications to handle the Fréchet algebra situation and the fact that all powers \( \{D^k \mid k \in \mathbb{N}\} \) of the derivation need to be considered.

**Lemma 2.4.** Let \( \mathcal{A} \) be a Fréchet algebra. Let \( \pi \in \text{Irred}(\mathcal{A}) \) be an irreducible representation of \( \mathcal{A} \) on an infinite dimensional complex vector space \( X \). Let

\[
\{\zeta_0, \zeta_1, \zeta_2, \ldots\}
\]

be a linearly independent (over \( \mathcal{A} \) ) set in \( X \). Then there exists \( a \in \mathcal{A} \) such that \( \pi(a)\zeta_0 = 0 \) and the set

\[
\{\pi(a)\zeta_1, \pi(a)\zeta_2, \pi(a)\zeta_3, \ldots\}
\]

is linearly independent in \( X \).

**Proof.** Let \( d \) be any translation invariant metric giving seminorm-wise convergence in \( \mathcal{A} \). For example, choose \( d(x, y) = \sum_{k=1}^{\infty} 2^{-k} (\|x - y\|_k)/(1 + \|x - y\|_k) \). Note that the metric \( d(\cdot, \cdot) \) is complete and that Cauchy sequences relative to \( d(\cdot, \cdot) \) are precisely the seminorm-wise Cauchy sequences. Choose, by Jacobson density, an element \( b_1 \in \mathcal{A} \) satisfying \( d(b_1, 0) < 2^{-1} \), \( \pi(b_1)\zeta_0 = 0 \), \( \{\pi(b_1)\zeta_1\} \) a linearly independent set in \( X \), and let \( c_1 = 0 \). We proceed by induction. Assume that \( b_1, b_2, \ldots, b_{k-1} \) have been chosen. Let \( c_k = b_1 + b_2 + \ldots + b_{k-1} \) and, again using Jacobson density, choose \( b_k \in \mathcal{A} \) satisfying \( d(b_k, 0) < 2^{-k} \), \( \pi(b_k)\zeta_0 = \pi(b_k)\zeta_1 = \ldots = \pi(b_k)\zeta_{k-1} = 0 \),
and \( \pi(b_k) \zeta_k \) not in the span of \( \{ \pi(c_k) \zeta_1, \pi(c_k) \zeta_2, \ldots, \pi(c_k) \zeta_k \} \). Continue by induction until all \( b_k \)'s have been chosen and let \( a = \sum_{i=1}^{\infty} b_i \), which converges.

We now use condition (1.3a) of Definition 1.3 twice. Since each \( b_i \in \text{ann}(\zeta_0) \) for \( i = 1, 2, \ldots \), it follows that \( \pi(a) \zeta_0 = 0 \). Let \( k \in \mathbb{N} \). Since \( a = \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{\infty} b_i \) and since each \( b_i \in \text{ann}(\zeta_k) \) for \( i > k \) it follows that

\[
\pi(a) \zeta_k = \pi(\sum_{i=1}^{k} b_i) \zeta_k = \sum_{i=1}^{k} \pi(b_i) \zeta_k = \pi(c_k) \zeta_k + \pi(b_k) \zeta_k
\]

Similarly, let \( j \in \mathbb{N} \) satisfy \( j < k \). Then

\[
\pi(a) \zeta_j = \sum_{i=1}^{j} \pi(b_j) \zeta_j = \sum_{i=1}^{k-1} \pi(b_j) \zeta_j = \pi(c_k) \zeta_j
\]

Since \( \pi(b_k) \zeta_k \) was chosen not to be in the span of \( \{ \pi(c_k) \zeta_1, \pi(c_k) \zeta_2, \ldots, \pi(c_k) \zeta_k \} \), it must be the case that \( \pi(a) \zeta_k \) is not in the span of \( \{ \pi(a) \zeta_1, \pi(a) \zeta_2, \ldots, \pi(a) \zeta_{k-1} \} \). This shows that \( \{ \pi(a) \zeta_1, \pi(a) \zeta_2, \ldots \} \) is linearly independent in \( X \), and finishes the proof of the lemma.

In all the following let \( D^0 \) denote the identity operator. As noted in Definition 1.1e, convergence in the Fréchet topology is equivalent to seminorm-wise convergence in each seminorm. When we say a function \( f \) is continuous with respect to a seminorm \( \| \cdot \|_q \) we mean that \( x_i \to x \) implies that \( \| f(x_i) - f(x) \|_q \to 0 \). The next lemma shows that either we can continue to take higher and higher powers of the derivation preserving continuity, or we get failure at some finite stage (note that \( k \) may equal one) which implies failure of continuity with respect to some seminorm.

**Lemma 2.5.** Let \( \mathcal{A} \) be a Fréchet algebra. Let \( \pi \in \text{Irred}(\mathcal{A}) \) be an irreducible representation of \( \mathcal{A} \) on an infinite dimensional complex vector space \( X \). Let \( k \in \mathbb{N} \). Let \( D \) be a derivation on \( \mathcal{A} \) such that the map \( x \to \pi(D^\ell x) \zeta \) is continuous for each \( \zeta \in X \) and \( \ell = 0, 1, 2, \ldots, k-1 \). Suppose further that for any fixed seminorm \( \| \cdot \|_q \) there exists a finite sequence \( q = s_0 \leq s_1 \leq s_2 \leq \ldots \leq s_{k-1} \) such that for each \( \ell \in \{ 0, 1, 2, \ldots, k-1 \} \) and for each \( \zeta \in X \) there exist positive constants \( \{ C_{\ell, \zeta} \}_{\ell=0}^{k-1} \) such that

\[
\| \pi(D^\ell x) \zeta \|_q \leq C_{\ell, \zeta} \| x \|_{s_\ell}
\]

for each \( x \in \mathcal{A} \). Then either
(i.) the map \(x \to \pi(D^k x)\zeta\) is discontinuous with respect to the seminorm \(\| \cdot \|_q\) for every non-zero vector \(\zeta \in X\) (i.e. there is always some \(x_i \to 0\) in \(A\) with \(\|\pi(D^k x_i)\zeta\|_q\) not converging to 0).

or

(ii.) there exists \(s_k \geq s_{k-1}\) and for each \(\zeta \in X\) a positive constant \(C_{k, \zeta}\) such that

\[\|\pi(D^k x)\zeta\|_q \leq C_{k, \zeta}\|x\|_{s_k}\]

for all \(x \in A\).

**Proof.** We note that Lemma 2.2 ensures that the zero-th case always yields continuity \((D^0 = I)\). Hence any failure of continuity would be at some \(k \in \mathbb{N}\). Suppose condition (i.) is false. Then there is some non-zero vector \(\zeta_0 \in X\) such that \(x \to \pi(D^k x)\zeta_0\) is continuous with respect to the seminorm \(\| \cdot \|_q\). Since this map is linear there must exist a positive constant \(C\) and \(s_k \in \mathbb{N}\) such that

\[\|\pi(D^k x)\zeta_0\|_q \leq C\|x\|_{s_k}\]

for all \(x \in A\). If it is not already the case, we can take \(s_k \geq s_{k-1}\). Clearly condition (ii.) holds for \(\zeta = 0\) so let \(\zeta\) be any other non-zero vector in \(X\). Choose, by Jacobson density, an element \(a \in A\) such that \(\pi(a)\zeta = \zeta\) and let \(\eta_\ell = \pi(D^{k-\ell} a)\zeta_0\) for \(\ell = 0, 1, \ldots, k - 1\). Letting \(x \in A\) we see that

\[
\|\pi(D^k x)\zeta\|_q = \|\pi(D^k x) a\zeta_0\|_q
\]

\[
= \|\pi(D^k (xa))\zeta_0 - \sum_{\ell=0}^{k-1} \binom{k}{\ell} \pi(D^\ell (xa) D^{k-\ell} (a))\zeta_0\|_q
\]

\[
\leq \|\pi(D^k (xa))\zeta_0\|_q + \sum_{\ell=0}^{k-1} \binom{k}{\ell} \|\pi(D^\ell (xa)) (\pi(D^{k-\ell} a)\zeta_0)\|_q
\]

\[
\leq C\|xa\|_{s_k} + \sum_{\ell=0}^{k-1} \binom{k}{\ell} C_{\ell, \eta_\ell}\|x\|_{s_\ell}
\]

\[
\leq \left( C\|a\|_{s_k} + \sum_{\ell=0}^{k-1} \binom{k}{\ell} C_{\ell, \eta_\ell} \right)\|x\|_{s_k}
\]

Hence, letting

\[C_{k, \zeta} = \left( C\|a\|_{s_k} + \sum_{\ell=0}^{k-1} \binom{k}{\ell} C_{\ell, \eta_\ell} \right)\]
we see that condition (ii.) must be true, completing the proof of the lemma.

**Lemma 2.6.** Let $\mathcal{A}$ be a Fréchet algebra. Let $\pi \in \text{Irred}(\mathcal{A})$ be an irreducible representation of $\mathcal{A}$ on an infinite dimensional complex vector space $X$. Let $D$ be a derivation on $\mathcal{A}$. Then for all $k \in \mathbb{N}$ and $\zeta \in X$ the map $x \to \pi(D^k x)\zeta$ is continuous.

**Proof.** If the result fails then it fails for some smallest $k \in \mathbb{N}$. Apply induction with the help of Lemma 2.5. There is then a seminorm $\| \cdot \|_q$ such that for every non-zero $\zeta \in X$ the map $x \to \pi(D^k x)\zeta$ is discontinuous with respect to $\| \cdot \|_q$ (i.e. there is always some $x_i \to 0$ in $\mathcal{A}$ with $\|\pi(D^k x_i)\zeta\|_q$ not converging to 0). In addition, there exist $q = s_0 \leq s_1 \leq s_2 \leq \ldots \leq s_{k-1}$ so that for each $\ell \in \{0, 1, 2, \ldots, k-1\}$ and for each $\zeta \in X$ there exist positive constants $\{C_{\ell, \zeta}\}_{\ell=0}^{k-1}$ such that

$$\|\pi(D^\ell x)\zeta\|_q \leq C_{\ell, \zeta} \|x\|_{s_\ell}$$

for each $x \in \mathcal{A}$. Note that if $k = 1$ then $\ell = 0$ and $s_0 = q$. For simplification of subscripts let $s = s_{k-1}$ in all of the following and note that $\| \cdot \|_{s_\ell} \leq \| \cdot \|_s$ for $\ell = 0, 1, \ldots, k-1$.

Since $X$ is infinite dimensional we can choose a linearly independent set of vectors $\{\zeta_i\}_{i=1}^\infty$ in $X$ satisfying $\|\zeta_i\|_q = 1$ for $i = 0, 1, 2, \ldots$. Using Lemma 2.4 inductively, and multiplying by non-zero scalars if necessary, construct a sequence $\{a_i\}_{i=1}^\infty$ in $\mathcal{A}$ such that the following three conditions hold:

(i.) $\pi(a_na_{n-1} \ldots a_1)\zeta_{n-1} = 0$ for $n \in \mathbb{N}$.

(ii.) the set

$$\{\pi(a_na_{n-1} \ldots a_1)\zeta_n, \pi(a_na_{n-1} \ldots a_1)\zeta_{n+1}, \pi(a_na_{n-1} \ldots a_1)\zeta_{n+2}, \ldots\}$$

is linearly independent in $X$.

(iii.) for $r = 1, 2, \ldots, n$ we have $\|a_na_{n-1} \ldots a_r\|_{s+n} < 2^{-n}$.

Fix $n \in \mathbb{N}$ temporarily. Let $\eta = \pi(a_na_{n-1} \ldots a_1)\zeta_n \neq 0$. Since $\pi(D^k(x)a_na_{n-1} \ldots a_1)\zeta_n = \pi(D^k(x))\eta$ it follows that the map

$$x \to \pi(D^k(x)a_na_{n-1} \ldots a_1)\zeta_n$$

is discontinuous with respect to the seminorm $\| \cdot \|_q$. Let

$$\eta_{\ell} = \pi(D^{k-\ell}(a_na_{n-1} \ldots a_1))\zeta_n$$

for $\ell = 0, 1, \ldots, k-1$. Then we see that

$$\pi(D^k(x)a_na_{n-1} \ldots a_1)\zeta_n = \pi(D^k(x)a_na_{n-1} \ldots a_1)\zeta_n - \sum_{\ell=0}^{k-1} \binom{k}{\ell} \pi(D^\ell x)\eta_{\ell}$$

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Since for \( \ell = 0, 1, \ldots, k - 1 \), we have that
\[
\|\pi(D^\ell x)\eta_\ell\|_q \leq C_{\ell, \eta_\ell} \|x\|_{s_\ell}
\]
for all \( x \in A \) it follows that the map
\[
x \to \pi(D^k(xa_n a_{n-1} \ldots a_1))\zeta_n \quad \text{(iv.)}
\]
must also be discontinuous with respect to the seminorm \( \| \cdot \|_q \).

Next consider the map
\[
x \to \pi(D^k(xa_{n+1} a_n \ldots a_1))\zeta_n
\]
Let \( \eta_\ell = \pi(D^{k-\ell}(a_{n+1} a_n \ldots a_1))\zeta_n \) for \( \ell = 0, 1, \ldots, k - 1 \). Then we see that
\[
\pi(D^k(xa_{n+1} a_n \ldots a_1))\zeta_n = 
\pi(D^k x)(\pi(a_{n+1} a_n \ldots a_1)\zeta_n) + \sum_{\ell=0}^{k-1} \binom{k}{\ell} \pi(D^\ell x)\pi(D^{k-\ell}(a_{n+1} a_n \ldots a_1))\zeta_n
\]
\[
= 0 + \sum_{\ell=0}^{k-1} \binom{k}{\ell} \pi(D^\ell x)\eta_\ell
\]
Since for \( \ell = 0, 1, \ldots, k - 1 \), we have that
\[
\|\pi(D^\ell x)\eta_\ell\|_q \leq C_{\ell, \eta_\ell} \|x\|_{s_\ell}
\]
for all \( x \in A \) it follows that there exists a positive constant \( C_n \) such that
\[
\|\pi(D^k(xa_{n+1} a_n \ldots a_1))\zeta_n\|_q \leq C_n \|x\|_s \quad \text{(v.)}
\]
for all \( x \in A \).

Hence, for each \( n \in \mathbb{N} \) we can choose \( x_n \in A \) satisfying the following two conditions:

(vi.) \( \|x_n\|_{s+n} \leq 1 \) and \( \|x_n\|_{s+n} \leq \min\{C^{-1}_j \mid j = 1, 2, \ldots, n\} \)

(vii.) \( \|\pi(D^k(x_n a_n a_{n-1} \ldots a_1))\zeta_n\|_q \geq n + \sum_{j=0}^{k-1} \|D^k(x_j a_j a_{j-1} \ldots a_1)\|_q \)

Let \( c = \sum_{j=1}^{\infty} x_j a_j a_{j-1} \ldots a_1 \), which converges absolutely in each seminorm due to conditions (iii.) and (vi.). Also, for each \( n \in \mathbb{N} \) let
\[
\alpha_n = x_{n+1} + \sum_{j=n+2}^{\infty} x_j a_j a_{j-1} \ldots a_{n+2}
\]
which also converges absolutely in each seminorm due to conditions (iii.) and (vi.).

Note that

\[
\|c_n\|_{s+n} = \|x_{n+1} + \sum_{j=n+2}^{\infty} x_j a_j a_{j-1} \cdots a_{n+2}\|_{s+n}
\]

\[
\leq \|x_{n+1}\|_{s+n+1} + \sum_{j=n+2}^{\infty} \|x_j\|_{s+j} 2^{-j}
\]

\[
\leq C_n^{-1} (1 + 1) = 2C_n^{-1}
\]

(viii.)

Note that

\[
c = \sum_{j=1}^{n-1} (x_j a_j a_{j-1} \cdots a_1) + (x_n a_n a_{n-1} \cdots a_1) + c_n (a_{n+1} a_n \cdots a_1)
\]

Since \(\|\zeta_n\|_q = 1\) for each \(n \in \mathbb{N}\) we see from Lemma 2.2 that

\[
\|D^k c\|_q \geq \|\pi(D^k c)\zeta_n\|_q
\]

\[
= \sum_{j=1}^{n-1} \pi(D^k (x_j a_j a_{j-1} \cdots a_1))\zeta_n + \pi(D^k (x_n a_n a_{n-1} \cdots a_1))\zeta_n
\]

\[
+ \pi(D^k (c_n (a_{n+1} a_n \cdots a_1)))\zeta_n
\]

\[
\geq \|\pi(D^k (x_n a_n a_{n-1} \cdots a_1))\zeta_n\|_q - \sum_{j=1}^{n-1} \|D^k (x_j a_j a_{j-1} \cdots a_1)\|_q \|\zeta_n\|_q
\]

\[
- \|\pi(D^k (c_n (a_{n+1} a_n \cdots a_1)))\zeta_n\|_q
\]

\[
\geq n - C_n \|c_n\|_s \geq n - C_n \|c_n\|_{s+n} \geq n - 2
\]

by conditions (v.), (vi.), (vii.) and (viii.). Since this holds for all \(n \in \mathbb{N}\) we have shown that \(\|D^k c\|_q = \infty\), a contradiction, and the result follows.

We finally obtain the desired result.

**Proposition 2.7.** Let \(A\) be a Fréchet algebra. Let \(\pi \in \text{Irred}(A)\) be an irreducible representation of \(A\) on an infinite dimensional complex vector space \(X\). Let \(P = \text{kernel}(\pi)\). Then for every \(k \in \mathbb{N}\) the map \(Q_P D^k\) is continuous and consequently \(\pi\) does not support the discontinuity of \(D\).

**Proof.** Apply lemma 2.6 and lemma 2.3. 

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§3. Finite dimensional irreducible representations of Fréchet Algebras.
From the previous section we know that for each $k \in \mathbb{N}$ the separating subspace $S(D^k)$ is contained in the intersection of the kernels of irreducible representations $\pi \in \text{Irred}(\mathcal{A})$ with $\dim(\pi) = \infty$. In this section we investigate where among the kernels of finite dimensional irreducible representations the discontinuity of a derivation could be localized. The example in [Read] shows that finite dimensional strictly irreducible representations can support the discontinuity of $D$ even in the commutative case. Note that if $P$ is the kernel of such a representation, Lemma 1.4iii. shows the equivalence of continuity of $Q_P D^k$ on $\mathcal{A}$ and continuity of its restriction to $P$.

The fact that the complex field is algebraically closed together with the Jacobson density theorem give us the following.

**Lemma 3.1.** Let $\mathcal{A}$ be a Fréchet algebra (over the complex field). Let $\pi \in \text{Irred}(\mathcal{A})$ with $\dim(\pi) < \infty$. If $P = \text{kernel}(\pi)$ we have that

(i.) $P$ has finite codimension, $\mathcal{A}/P \cong \mathcal{M}_{\dim(\pi)}$, and

(ii.) $\mathcal{A}/P$ is a Banach algebra under one of its quotient seminorms.

**Proof.** By the assumption that $\dim(\pi) < \infty$ and the Jacobson density theorem, we know that $\mathcal{A}/P$ is isomorphic to a full matrix algebra with entries in a finite-dimensional division complex algebra $\mathcal{D}$. Since the complex field $\mathbb{C}$ is algebraically closed, we have in fact $\mathcal{D} = \mathbb{C}$. This proves (i.).

Note that $\mathcal{A}/P$ is simple. A submultiplicative seminorm on a simple algebra is either zero or a norm, thus proving (ii.).

As a consequence of Lemma 3.1(ii.) we will simply use $\| \cdot \|$ to denote the Banach algebra norm on $\mathcal{A}/P$.

We are interested in how often the singular case can occur. Again, let $D^0$ denote the identity operator. If $Q_P D^k$ is discontinuous for some $k \in \mathbb{N}$ then there is a first such $k = k_0 \in \mathbb{N}$ and $Q_P D^\ell$ is continuous from $\mathcal{A}$ to $\mathcal{A}/P$ for $\ell = 0, 1, \ldots, (k_0 - 1)$.

Furthermore, since $\mathcal{A}/P$ is finite dimensional and isomorphic to an algebra of matrices there is a smallest $m_0 \in \mathbb{N}$ such that

$$\{ (\mathcal{A}/P, \| \cdot + P \|_m) \mid m \geq m_0 \}$$
is a set of continuously isomorphic Banach algebras. Hence there is some $q_0 \in \mathbb{N}$ and a constant $C_m > 0$ for $m = m_0, m_0 + 1, \ldots$ such that for all $x \in \mathcal{A}$ we have

$$\|Q_P D^\ell(x)\|_m \leq C_m \|x\|_{q_0}$$

for $m \geq m_0$ and $\ell = 0, 1, \ldots, (k_0 - 1)$. We formalize this as a definition.

**Definition 3.2.** Let $D$ be a derivation on a Fréchet algebra $\mathcal{A}$. Let $\pi \in \text{Irred}(\mathcal{A})$ be an irreducible representation of $\mathcal{A}$ on an finite dimensional complex vector space $X$ and let $P = \ker(\pi)$. Suppose $\pi$ supports the discontinuity of $D$ (equivalently, $D$ is singular at $P$). Let $k$ be the minimum natural number such that $Q_P D^k$ is discontinuous. Let $q$ be the smallest index for which $Q_P D^\ell$ for $\ell = 0, 1, \ldots, (k - 1)$ is continuous as a linear map from $(\mathcal{A}, \|\cdot\|_q)$ into $\mathcal{A}/P$. We call $q$ the index of $P$ and we write $q = \text{index}(P)$.

Finally, for each $q \in \mathbb{N}$ define the $q$th-exceptional set $\mathcal{E}_q(D)$ as follows

$$\mathcal{E}_q(D) \equiv \{P \in \text{Prim}(\mathcal{A}) \mid D \text{ is singular at } P \text{ and } \text{index}(P) \leq q\}$$

Note that this definition does not depend on the choice of norm on $\mathcal{A}/P$ since this Banach algebra is finite dimensional.

Our goal is to show that each $\mathcal{E}_q(D)$ is a finite (or empty) set, thereby reducing the problem to dealing with countably many points at which $D$ is singular. This result is well known in the case of a Banach algebra since there is only one seminorm (see [Thomas3, Proposition 1.10] which is strongly indebted to [Johnson-Sinclair]). It is also well known in the case of a Jordan-Banach algebra (see [Villena, Theorem 7]). The generalization to Fréchet algebras requires using the somewhat weaker conclusions of the automatic continuity for linear functions on Fréchet spaces (see [Thomas1]), but this is sufficient.

We proceed with the following two lemmas which are essentially due to Johnson and Sinclair [Johnson-Sinclair, Lemma 3.1 and Lemma 3.2] again with modifications to handle the Fréchet algebra situation.

**Lemma 3.3.** (Johnson and Sinclair) Let $\{\pi_i\}_{i=1}^j \subseteq \text{Irred}(\mathcal{A})$ be non-equivalent strictly irreducible representations of a Fréchet algebra $\mathcal{A}$ over the complex field on (respectively) finite dimensional complex vector spaces $\{X_i\}_{i=1}^j$. Let $\dim(X_i) = n_i$, let $I_i$ be the identity matrix in $\mathcal{M}_{n_i}$, and let $P_i = \ker(\pi_i)$ for
\( i = 1, 2, \ldots, j \). Then

\[
\mathcal{A}/(P_1 \cap P_2 \cap \ldots \cap P_j) \cong (\mathcal{A}/P_1) \oplus (\mathcal{A}/P_2) \oplus \ldots \oplus (\mathcal{A}/P_j)
\]

\[
\cong \mathcal{M}_{n_1} \oplus \mathcal{M}_{n_2} \oplus \ldots \oplus \mathcal{M}_{n_j},
\]

and there exists \( a \in \mathcal{A} \) such that \( \pi_i(a) = 0 \) for \( i < j \) but \( \pi_j(a) = I_j \).

**Proof.** Although [Johnson-Sinclair, Lemma 3.1] is stated for Banach algebras, the proof goes through verbatim for Fréchet algebras due to the cofiniteness of the primitive ideals. Therefore the assertion of the isomorphism above holds and the isomorphism can be implemented via the map

\[
\varphi: a \to (\pi_1(a), \pi_2(a), \ldots, \pi_j(a))
\]

Hence there exists an element \( a_0 \in \mathcal{A} \) with \( \pi_i(a_0) = 0 \) for \( i < j \) but \( \pi_j(a_0) = I_j \). It is clear that kernel(\( \varphi \)) = \( P_1 \cap P_2 \cap \ldots \cap P_j \) and that kernel(\( \varphi \)) is closed since the primitive ideals \( P_i \) are closed for \( i = 1, 2, \ldots, j \) (here we are using the fact that each \( \pi_i \in \text{Irred}(\mathcal{A}) \)). \( \blacksquare \)

**Lemma 3.4.** (Johnson and Sinclair) Let \( \{\pi_i\}_{i=1}^\infty \in \text{Irred}(\mathcal{A}) \) be non-equivalent strictly irreducible representations of a Fréchet algebra \( \mathcal{A} \) over the complex field on (respectively) finite dimensional complex vector spaces \( X_i \), for all \( i \in \mathbb{N} \). Then there exists a sequence \( \{a_i\}_{i=1}^\infty \subseteq \mathcal{A} \) such that

\[
\pi_n(a_m) = 0
\]

whenever \( m > n \), but

\[
\pi_n(a_m) \text{ is regular in } \mathcal{M}_{\text{dim}(X_n)} \text{ for } n \geq m
\]

**Proof.** Let \( d \) be any translation invariant metric giving seminorm-wise convergence in \( \mathcal{A} \). For example, choose \( d(x, y) = \sum_{k=1}^\infty 2^{-k}(\|x - y\|_k)/(1 + \|x - y\|_k) \). Let \( \text{dim}(X_i) = n_i \) for \( i \in \mathbb{N} \). Let \( I_i \) denote the identity matrix in \( \mathcal{M}_{n_i} \). Temporarily fix \( k \in \mathbb{N} \). Choose \( z_j \) for \( j = k, k + 1, \ldots \) such that \( \pi_i(z_j) = 0 \) for \( 1 \leq i < j \) and \( \pi_j(z_j) = I_j \). Lemma 3.3 guarantees that such \( z_j \) exist for \( j = k, k + 1, \ldots \). Next choose a sequence of of positive reals \( \{\epsilon_j\}_{j=k}^\infty \) by induction so that the two conditions

\[
d(0, \epsilon_j z_j) < 2^{-j} \text{ and}
\]

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\[ \pi_j(\epsilon_k z_k + \ldots + \epsilon_j z_j) = \pi_j(\epsilon_k z_k + \ldots + \epsilon_{j-1} z_{j-1}) + \epsilon_j I_j \text{ is regular} \]

are satisfied. This is possible because

\[ \pi_j(\epsilon_k z_k + \ldots + \epsilon_{j-1} z_{j-1}) + \lambda I_j \]

is singular in \( \mathcal{M}_{n_j} \) for only a finite number of values of \( \lambda \). It is routine to check that \( a_k = \sum_{j=k}^{\infty} \epsilon_j z_j \) has the necessary properties. \( \blacksquare \)

**Proposition 3.5.** Let \( D \) be a (possibly discontinuous) derivation on a Fréchet algebra \( \mathcal{A} \). Then for each \( q \in \mathbb{N} \) the exceptional set \( \mathcal{E}_q(D) \) is either empty or a finite set of primitive ideals of finite codimension.

**Proof.** Suppose that the result fails for some \( q \in \mathbb{N} \). For each primitive ideal \( P \in \text{Prim}(\mathcal{A}) \) of finite codimension fix a quotient norm on \( \mathcal{A}/P \) (since all such norms are equivalent) and denote it by \( \| \cdot + P \| \) (with no subscript). We can then find a sequence of non-equivalent strictly irreducible representations \( \{ \pi_i \}_{i=1}^{\infty} \subseteq \text{Irred}(\mathcal{A}) \) with primitive ideals \( \{ P_i = \text{kernel}(\pi_i) \}_{i=1}^{\infty} \subseteq \text{Prim}(\mathcal{A}) \) such that there exists \( k_i \in \mathbb{N} \) and \( C_i > 0 \) such that

\[ \| D^\ell(x) + P_i \| \leq C_i \| x \|_q , \]

for all \( x \in \mathcal{A} \) and \( \ell < k_i \), but

\[ Q_{P_i}D^{k_i} \text{ is discontinuous} \]

for \( i = 1, 2, \ldots \). Apply Lemma 3.4 in order to obtain the sequence \( \{ a_i \}_{i=1}^{\infty} \). Let \( x \in \mathcal{A} \) and compute

\[
Q_{P_n}D^{k_n}(xa_m \ldots a_2a_1) = (Q_{P_n}D^{k_n}(x))(Q_{P_n}(a_m \ldots a_2a_1)) \\
+ \sum_{\ell=0}^{k_n-1} {k_n \choose \ell} (Q_{P_n}D^\ell(x))(Q_{P_n}D^{k_n-\ell}(a_m \ldots a_2a_1))
\]

for \( m, n \in \mathbb{N} \). Note that the map \( x \rightarrow Q_{P_n}D^\ell(x) \) is continuous for \( \ell = 0, 1, \ldots, (k_n - 1) \) from \( (\mathcal{A}, \| \cdot \|_q) \) into \( \mathcal{A}/P_n \). If \( m > n \) we have that \( \pi_n(a_m) = 0 \) so \( Q_{P_n}(a_m \ldots a_2a_1) = 0 \) and the map \( x \rightarrow Q_{P_n}D^{k_n}(xa_m \ldots a_2a_1) \) is continuous from \( (\mathcal{A}, \| \cdot \|_q) \) into \( \mathcal{A}/P_n \). In particular, we can find a constant \( C_n > 0 \) such that both

\[ \| Q_{P_n}D^{k_n-1}y \| \leq C_n \| y \|_q \]
and
\[ \|Q_{P_n}D^{k_n}(ya_{n+1}a_n \ldots a_2a_1)\| \leq C_n \|y\|_q \]

hold for all \( y \in \mathcal{A} \). However, if \( m = n \) since \( \pi_n(a_n \ldots a_2a_1) \) is regular in \( \mathcal{M}_{\dim(\pi_n)} \) and since the map \( x \rightarrow Q_{P_n}D^{k_n}(x) \) is discontinuous, it must be the case that the map \( x \rightarrow Q_{P_n}D^{k_n}(xa_n \ldots a_2a_1) \) is discontinuous.

For each \( n \in \mathbb{N} \) use the discontinuity of the map \( x \rightarrow Q_{P_n}D^{k_n}(xa_n \ldots a_2a_1) \) in order to choose \( x_n \in \mathcal{A} \) so that \( \|x_n\|_{q+n} < 2^{-n} \), and \( \|x_n a_n \ldots a_{r+1}a_r\|_{q+n} < 2^{-n} \) for all \( r = 1,2,\ldots,n \), but

\[ \|D^{k_n}(x_na_n \ldots a_2a_1) + P_n\| > nC_n + \|D^{k_n}\left(\sum_{i=1}^{n-1}x_ia_i \ldots a_2a_1\right) + P_n\| \]

where the constant \( C_n \) is the one above. Let \( x = \sum_{i=1}^\infty x_ia_i \ldots a_2a_1 \in \mathcal{A} \) which is easily seen to converge. For each \( n \in \mathbb{N} \) we can obtain a lower bound for \( \|Dx\|_q \) as follows:

\[ \|Dx\|_q \geq C_n^{-1}\|(Q_{P_n}D^{k_n-1})(Dx)\| \]

\[ = C_n^{-1}\|D^{k_n}x + P_n\| \]

\[ = C_n^{-1}\|D^{k_n}\left(\sum_{i=1}^\infty x_ia_i \ldots a_2a_1\right) + P_n\| \]

\[ \geq C_n^{-1}\left(\|D^{k_n}(x_na_n \ldots a_2a_1) + P_n\| - \|D^{k_n}\left(\sum_{i=1}^{n-1}x_ia_i \ldots a_2a_1\right) + P_n\| \right. \]

\[ \quad - \left. \|D^{k_n}\left(\sum_{i=n+1}^\infty x_ia_i \ldots a_2a_1\right) + P_n\| \right) \]

\[ \geq C_n^{-1}\left(nC_n - \|D^{k_n}\left((x_{n+1} + \sum_{i=n+2}^\infty x_ia_i \ldots a_{n+2})a_{n+1} \ldots a_2a_1\right) + P_n\| \right) \]

Since \( \|(\sum_{i=n+1}^\infty x_ia_i \ldots a_{n+2})\|_q \leq 1 \) we have that

\[ \|Dx\|_q \geq C_n^{-1}\left(nC_n - C_n\right) = (n-1) \]

for all \( n \in \mathbb{N} \), a contradiction. This establishes the proof of the proposition. \( \blacksquare \)

As noted previously, taking into account (countable) Fréchet products of the counterexample in [Read], we see that we have proved the best possible result.
which establishes a limit for the discontinuity of $D$ and is expressed in terms of those strictly irreducible representations which are compatible with the algebra’s topology.

In order to go further we need to specialize to commutative Fréchet algebras $\mathcal{A}$ since we need some topology on $\text{Irred}(\mathcal{A})$ and, in the commutative case, we at least have the topology of pointwise convergence (which is the weak* topology in the case of Banach algebras) as functions on $\mathcal{A}$. We will continue to call this the weak* topology, even though it is not in general a locally compact topology on $\text{Irred}(\mathcal{A})$ when $\mathcal{A}$ is a Fréchet algebra. Since $\mathcal{A}$ is commutative, it is clear that all elements $\pi \in \text{Irred}(\mathcal{A})$ with $\dim(\pi) < \infty$ are multiplicative homomorphisms into the complex field.

**Lemma 3.6.** Suppose that $T$ and $\{T_i\}_{i=1}^{\infty}$ are continuous linear operators from a Fréchet space $(X, \| \cdot \|_{n})$ to a Banach space $(Y, \| \cdot \|)$ satisfying

$$\lim_{i \to \infty} T_i x = Tx$$

for all $x \in X$. Then there exist $p \in \mathbb{N}$ and $C \in \mathbb{R}^+$ such that

$$\|T_i x\| \leq C \|x\|_p$$

for all $x \in X$ and $i \in \mathbb{N}$.

**Proof.** Since $T$ is continuous there exist $q \in \mathbb{N}$ and $K \in \mathbb{R}^+$ such that $\|T x\| \leq K \|x\|_q$ for all $x \in X$. Since $\lim_{i \to \infty} T_i x = Tx$ for every $x \in X$ there exists $K_x$ such that $\|T_i x\| \leq K_x \|x\|_q$ for every $i \in \mathbb{N}$. Let $X_n = \{x \in X \mid K_x \leq n\}$. Then $\cup_{n=1}^{\infty} X_n = X$ and since $\|T_i x\| \leq n \|x\|_q$ is a closed condition, each $X_n$ is closed. Since $(X, \| \cdot \|_{n})$ is a complete metric space there is some $X_N$ which contains an $(\epsilon, p)$-ball

$$B(x_0, \epsilon, p) = \{x \in X \mid \|x - x_0\|_p < \epsilon\} \subseteq X_N$$

and, by passing to a somewhat larger $N$, we may assume that $x_0 \in X_N$. Let $x$ be any element of $X$ with $\|x\|_p = \epsilon/2$. Then $\|(x_0 + x) - x_0\|_p < \epsilon$ so

$$\|T_i x\| = \|T_i (x_0 + x) - T_i x_0\| \leq N \|x_0 + x\|_p + N \|x_0\|_p \leq 2N \left(1 + \frac{\|x_0\|_p}{\epsilon}\right) \frac{\epsilon}{2}$$

Hence, for any $x \in X$ we have $\|T_i x\| \leq C \|x\|_p$ where

$$C = 2N \left(1 + \frac{\|x_0\|_p}{\epsilon}\right)$$
This finishes the proof of the lemma.

**Lemma 3.7.** Let $D$ be a (possibly discontinuous) derivation on a commutative Fréchet algebra $\mathcal{A}$. Then the union of the exceptional sets $\bigcup_{q=1}^{\infty} \mathcal{E}_q(D)$ consists only of sequentially isolated points in the weak* topology.

**Proof.** Suppose that the result fails and there exists some pairwise disjoint sequence $\{\pi_i\}_{i=1}^{\infty} \subseteq \bigcup_{q=1}^{\infty} \mathcal{E}_q(D)$ and $\pi \in \bigcup_{q=1}^{\infty} \mathcal{E}_q(D)$ such that

$$\lim_{i \to \infty} \pi_i y = \pi y$$

for each $y \in \mathcal{A}$. Let $P = \text{kernel}(\pi)$. There exist $k, q \in \mathbb{N}$ such that $Q_P D^k$ is discontinuous, but

$$Q_P D^\ell : (\mathcal{A}, \| \cdot \|_q) \to \mathcal{A}/P \cong \mathbb{C}$$

is continuous for $\ell = 0, 1, 2, \ldots k - 1$. For each $i \in \mathbb{N}$ let $P_i = \text{kernel}(\pi_i)$. For each $i \in \mathbb{N}$ there exist $k_i, q_i \in \mathbb{N}$ such that $Q_{P_i} D^{k_i}$ is discontinuous, but

$$Q_{P_i} D^\ell : (\mathcal{A}, \| \cdot \|_{q_i}) \to \mathcal{A}/P_i \cong \mathbb{C}$$

is continuous for $\ell = 0, 1, 2, \ldots k_i - 1$. We first claim that eventually $k_i \leq k$. If not, drop to a subsequence such that $k_i > k$ for all $i \in \mathbb{N}$. Then for all $i \in \mathbb{N}$ it would be the case that $Q_{P_i} S(D^k) = \{0\}$, and, hence, that $\pi_i(S(D^k)) = \{0\}$. Since

$$\lim_{i \to \infty} \pi_i z = \pi z$$

for all $z \in S(D^k)$ this implies that $\pi(S(D^k)) = \{0\}$, or, equivalently, that $S(D^k) \subseteq \text{kernel}(\pi) = P$ so that $Q_P D^k$ is continuous, a contradiction.

Therefore, without loss of generality, we may assume that for all $i \in \mathbb{N}$ we have $k_i \leq k$ and, since each $\mathcal{E}_q(D)$ is a finite set, that $\lim_{i \to \infty} \text{index}(P_i) = +\infty$. For each $i \in \mathbb{N}$ choose $m_i \in \{0, 1, 2, \ldots, k_i - 1\} \subseteq \{0, 1, 2, \ldots, k - 1\}$ such that

$$Q_{P_i} D^{m_i} : (\mathcal{A}, \| \cdot \|_{\text{index}(P_i)-1}) \to \mathcal{A}/P_i \cong \mathbb{C}$$

is discontinuous. Since there are infinitely many $i$ but only finitely many choices for each $m_i$ we can drop to a subsequence of $\{\pi_i\}_{i=1}^{\infty}$ which uses $m_i = m \in \{0, 1, 2, \ldots, k - 1\}$ exclusively. That is, we can assume without loss of generality that there is $m \in \{0, 1, 2, \ldots, k - 1\}$ such that

$$Q_P D^m : (\mathcal{A}, \| \cdot \|_h) \to \mathcal{A}/P_i \cong \mathbb{C}$$

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is eventually discontinuous as \( i \to \infty \) for each fixed \( h \in \mathbb{N} \) (although each \( Q_{P_i} D^m \) is certainly continuous with respect to the Fréchet space topology generated by all seminorms).

Let \( T_i = \pi_i D^m \) for \( i \in \mathbb{N} \) and let \( T = \pi D^m \). It is clear that \( \lim_{i \to \infty} T_i x = T x \) for each fixed \( x \in \mathcal{A} \). Since \( Q_P D^m \) is continuous if and only if \( S(D^m) \subseteq P \) if and only if \( \pi(S(D^m)) = \{0\} \) if and only if \( \pi D^m \) is continuous, and since a similar argument applies for each \( Q_{P_i} D^m \), it is clear that \( T \) and \( \{T_i\}_{i=1}^{\infty} \) are continuous linear operators from the Fréchet space \( \mathcal{A} \) to the Banach space \( \Phi \). The previous lemma implies that there exists \( p \in \mathbb{N} \) and \( C \in \mathbb{R}^+ \) such that

\[
\|T_i x\| = |\pi_i D^m x| \leq C \|x\|_p
\]

for all \( x \in \mathcal{A} \) and \( i \in \mathbb{N} \). But then

\[
Q_{P_i} D^m : (\mathcal{A}, \|\cdot\|_p) \to \mathcal{A}/P_i \cong \Phi
\]

is continuous for all \( i \in \mathbb{N} \), a contradiction. Hence no such sequence \( \{\pi_n\}_{i=1}^{\infty} \) exists, ending the proof of the lemma.

An immediate corollary is

**Lemma 3.8.** Let \( D \) be a (possibly discontinuous) derivation on a commutative Fréchet algebra \( \mathcal{A} \) whose structure space \( \Phi_\mathcal{A} \) is a compact metric space in the weak* topology. Then the union of the exceptional sets \( \cup_{i=1}^{\infty} \mathcal{E}_q(D) \) is a finite set and there exists \( q_0 \) such that \( \cup_{i=1}^{\infty} \mathcal{E}_q(D) = \mathcal{E}_{q_0}(D) \).

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References


