Local Power Series Quotients of Commutative Banach and Fréchet Algebras

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Abstract. We consider the relationship between derivations and local power series quotients for a locally multiplicatively convex Fréchet algebra (this includes the case of a Banach algebra). In §2 we derive necessary conditions for a commutative Fréchet algebra to have a local power series quotient. Our main result here is Proposition 2.6, which shows that if the generating element has finite closed descent, the algebra cannot be simply a radical algebra with identity adjoined – it must have non-trivial representation theory; if the generating element does not have finite closed descent then the algebra cannot be a Banach algebra, and the generating element must be locally nilpotent (but non-nilpotent) in an associated quotient algebra. In §3 we impose some additional conditions which are automatic for a Banach algebra but required in the case of a Fréchet algebra in order to use standard techniques from representation theory. We consider with which strictly irreducible representations the discontinuity of a derivation must be associated. The main result in this section is Proposition 3.14, which shows that when consideration is fixed upon a single seminorm, the exceptional set of irreducible representations supporting the discontinuity must be a finite set. We also prove that derivations on commutative Fréchet algebras whose structure spaces are compact metric in the weak* topology have only finitely many such exceptional points overall. This leads naturally to the case in §4 where we consider a derivation $D$ on a commutative radical Fréchet algebra $\mathcal{R}^d$ with identity adjoined. We show in Theorem 4.8 that a derivation $D$ whose discontinuity is not concentrated in the (Jacobson) radical forces $\mathcal{R}^d$ to have a local power series quotient. The question whether such a derivation can have a separating ideal so large it actually contains the identity element has been recently settled in the affirmative by C. J. Read.

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§1. Introduction. The Singer-Wermer conjecture states that a (possibly discontinuous) derivation on a commutative Banach algebra maps into the (Jacobson) radical. In establishing a proof of this conjecture one first uses reductions of B. E. Johnson [Johnson] and Thomas [Thomas2] to show that if the conjecture fails some commutative radical Banach algebra $\mathcal{A}$ has an element $x$ with finite closed descent (see Definition 1.13) and a formal power series quotient based at that $x$ (see Definition 2.1). Such a quotient is then shown to be impossible by constructing a finite subset called a recalcitrant system (see [Thomas2, Definition 3.3]). Recalcitrant systems essentially preclude prime-like properties, but they seem to require either finite closed descent or a related condition such as $a^n \in \overline{a^nA}$ for construction.

Much the same strategy can be used to analyze a derivation on a locally multiplicatively convex Fréchet algebra, and we will do this in Section 3. What is surprising is that the conditions necessary for the existence of a formal power series quotient are not greatly different whether one considers a Banach or Fréchet algebra (see Proposition 2.6) although such quotients are far more prevalent in Fréchet algebras where is possible to have locally nilpotent, non-nilpotent, elements.

The Fréchet and Banach algebras in this paper will generally be commutative. Commutativity is essential in Section 2 since the kernel of $\theta$ (see Definition 2.1) needs to be an intersection of range spaces. The non-commutative version of Definition 2.1 requires handling the ideal generated by the commutators and this is probably one contributing reason why the non-commutative Singer-Wermer conjecture has remained unproven. However, in the reduction results in Section 3 we cover the more general non-commutative case since it can be handled in the same way as the commutative case.

Throughout the paper we will let $\mathcal{A}$ denote a (locally multiplicatively convex) Fréchet algebra over the complex field. It is convenient for our purposes to have an identity element 1. If $\mathcal{A}$ already has an identity element we will let $\mathcal{A}^1$ denote $\mathcal{A}$. Otherwise we will adjoin an identity in the usual way so that $\mathcal{A}^1 \cong \mathbb{C} 1 \oplus \mathcal{A}$. We will assume without loss of generality that the locally convex topology $\tau$ on $\mathcal{A}$ arises from a countable family of increasing submultiplicative seminorms $\{\| \cdot \|_n \mid n \in \mathbb{N}\}$ satisfying

\[(1.1a) \text{ for all } a \in \mathcal{A} \|a\|_n \leq \|a\|_m \text{ whenever } n \leq m \text{ in } \mathbb{N}, \text{ and } \|a\|_n = 0 \text{ for all } n \in \mathbb{N} \text{ if and only if } a = 0.\]
(1.1b) \( \| \lambda a \|_n = |\lambda| \| a \|_n \) for all \( a \in \mathbb{A} \), \( \lambda \in \mathbb{C} \), and \( n \in \mathbb{N} \).

(1.1c) \( \|a + b\|_n \leq \|a\|_n + \|b\|_n \) for all \( a, b \in \mathbb{A} \) and \( n \in \mathbb{N} \).

(1.1d) \( \|ab\|_n \leq \|a\|_n \|b\|_n \) for all \( a, b \in \mathbb{A} \) and \( n \in \mathbb{N} \).

(1.1e) \( \lim_{i \to \infty} a_i = a \) in the Fréchet topology \( \tau \) of \( \mathbb{A} \) if and only if for each fixed \( n \in \mathbb{N} \) we have that \( \lim_{i \to \infty} \|a - a_i\|_n = 0 \).

If \( \mathbb{A} \) does not have an identity element we extend each of these seminorms to \( \mathbb{A}^\mathbb{C} \) in the usual way, that is

\[
\|\lambda 1 + a\|_n = |\lambda| + \|a\|_n
\]

for \( \lambda \in \mathbb{C} \) and \( a \in \mathbb{A} \). It is routine to check that (1.1a–e) still hold. We will work primarily with \( \mathbb{A}^\mathbb{C} \) in the following. In the very special case that the Fréchet Algebra \( \mathbb{A} \) is actually a Banach algebra then eventually each seminorm depends continuously on the previous one.

Throughout this paper we use the terms algebra, ideal, and subspace, in the algebraic sense, that is, we do not assume these substructures are closed.

A commutative Fréchet algebra \( \mathbb{A}^\mathbb{C} \) can be realized as an inverse (or projective) limit via the Arens-Michael isomorphism (see [Arens] and [Michael]) in the following standard way. For each \( n \in \mathbb{N} \) define

\[
\mathcal{I}_n \equiv \{ a \in \mathbb{A}^\mathbb{C} \mid \|a\|_n = 0 \}
\]

It is routine to check that \( \mathcal{I}_n \) is a closed ideal of \( \mathbb{A}^\mathbb{C} \) and that \( (\mathbb{A}^\mathbb{C}/\mathcal{I}_n , \| \cdot \|_n) \) is a commutative normed linear algebra over the complex field. However, it is not necessarily complete. Let \( \mathcal{B}_n \) denote its completion in the \( \| \cdot \|_n \) seminorm (which, on \( \mathbb{A}^\mathbb{C}/\mathcal{I}_n \) is actually a norm). We will let \( \| \cdot \|_{\mathcal{B}_n} \) denote the extension of \( \| \cdot \|_n \) to the completion. Since \( \mathbb{A}^\mathbb{C} \) is unital, it is routine to check that \( \mathcal{B}_n \) is a commutative unital Banach algebra. Let \( \pi_n \) denote the canonical continuous algebra homomorphism from \( \mathbb{A}^\mathbb{C} \) into \( \mathcal{B}_n \), with kernel \( \mathcal{I}_n \). It is clear that \( \pi_n(\mathbb{A}^\mathbb{C}) \) is dense in \( \mathcal{B}_n \). Since \( \| \cdot \|_{n+1} \) is stronger than \( \| \cdot \|_n \) it is also clear that \( \mathcal{I}_{n+1} \subseteq \mathcal{I}_n \) and there is a natural continuous algebra homomorphism \( \iota_n \) from \( (\mathbb{A}^\mathbb{C}/\mathcal{I}_{n+1} , \| \cdot \|_{n+1}) \) into \( (\mathbb{A}^\mathbb{C}/\mathcal{I}_n , \| \cdot \|_n) \). This gives rise to a continuous algebra homomorphism from \( \mathbb{B}_{n+1} \) into \( \mathcal{B}_n \) for which we will use the same symbol, \( \iota_n \). The standard structure theorem for commutative Fréchet algebras [Michael, Theorem 5.1] states that \( \mathbb{A}^\mathbb{C} \) is isomorphic to the inverse (or projective) limit

\[
\lim_{n \to \infty} (\mathbb{B}_n , \iota_n)
\]

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and that the $n$-th coordinate projections are essentially realizable as the $\pi_n$’s.

We give two examples here in order to make these ideas clearer.

**Example 1.2.** Let $\mathcal{A} = L^1_{loc}(\mathbb{R}^+)$ where

$$L^1_{loc}(\mathbb{R}^+) \equiv \{ f \text{ measurable on } [0, \infty) \mid \int_0^n |f(x)| \, dx < \infty \text{ for all } n \in \mathbb{N} \}$$

and multiplication is given by convolution, that is

$$f * g(t) = \int_0^t f(t - s)g(s) \, ds$$

For each $n \in \mathbb{N}$ we may let

$$\|f\|_n = \int_0^n |f(t)| \, dt$$

and it is routine to verify that (1.1a–e) hold. Note that

$$\mathcal{I}_n = \{ f \text{ measurable on } [0, \infty) \mid f = 0 \text{ a.e. on } [0, n] \}$$

Therefore, $\mathcal{B}_n = L^1([0, n])$, a commutative radical Banach algebra for each $n \in \mathbb{N}$ and each $\pi_n$ is onto $\mathcal{B}_n$.

**Example 1.3.** Let $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R})$ where

$$\mathcal{C}^\infty(\mathbb{R}) \equiv \{ f \text{ continuous on } (-\infty, \infty) \mid f^{(n)} \text{ exists for each } n \in \mathbb{N} \}$$

with pointwise multiplication. This commutative Fréchet algebra is unital. Since the Leibniz formula gives us

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)}g^{(i)}$$

we will get a submultiplicative seminorm if we define

$$\|f\|_n = \sup_{-n \leq x \leq n} \sum_{i=0}^{n} \frac{|f^{(i)}(x)|}{i!}$$

for each $n \in \mathbb{N}$. Since only the first $n$ derivatives are used to establish the $n$-th seminorm is routine to check that

$$\mathcal{B}_n = \mathcal{C}^n([-n, n])$$
Since there exist functions which are $n$ times continuously differentiable but not $n+1$ times continuously differentiable we see that the image of $\pi_n$ is dense but not onto $\mathcal{B}_n$.

We now require some standard definitions concerning nilpotency, spectrum, and torsion.

**Definition 1.4.** Let $\mathcal{A}$ be a Fréchet algebra. We say that an element $a \in \mathcal{A}$ is nilpotent if there exists $n \in \mathbb{N}$ such that $a^n = 0$. We say that $a$ is locally nilpotent if for each $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\|a^n\|_N = 0$. If this is not the case, there must exist some $N \in \mathbb{N}$ such that $\|a^n\|_N > 0$ for all $n \in \mathbb{N}$ and we say that $a$ is non-locally nilpotent.

If $\mathcal{A}$ is actually a Banach algebra, local nilpotency and nilpotency are equivalent.

**Definition 1.5.** We define the spectrum of an element $a \in \mathcal{A}$ in the usual way

$$\sigma(a) \equiv \{ \lambda \in \mathbb{C} \mid (\lambda I - a) \text{ is not invertible in } \mathcal{A} \}$$

and the spectral radius in the usual way

$$\rho_{\sigma}(a) \equiv \sup \{ |\lambda| \mid \lambda \in \sigma(a) \}$$

Since the spectrum in a Fréchet algebra (unlike the case of a Banach algebra) need not be compact, we allow $\rho_{\sigma}(a)$ to take the value $+\infty$.

**Definition 1.6.** Let $\mathcal{A}$ be a commutative Fréchet algebra and let $a \in \mathcal{A}$. For each $n \in \mathbb{N}$ the element $\pi_n(a)$ is an element in the Banach algebra $\mathcal{B}_n$ so that $\sigma(\pi_n(a))$ is a non-empty compact subset of the complex plane. For ease of notation we denote $\sigma_n(a) = \sigma(\pi_n(a))$ for all $n \in \mathbb{N}$.

It is routine to check that $\sigma_n(a) \subseteq \sigma_{n+1}(a)$ and that

$$\max \{ |\lambda| \mid \lambda \in \sigma_n(a) \} = \lim_{i \to \infty} \|\pi_n(a^i)\|_{\mathcal{B}_n}^{1/i} \leq \|\pi_n(a)\|_{\mathcal{B}_n} = \|a\|_n$$

for all $n \in \mathbb{N}$. It is also the case that $\sigma(a) = \bigcup_{n=1}^{\infty} \sigma_n(a)$ (see [Michael, Theorem 5.3]).

**Definition 1.7.** Let $\mathcal{A}$ be a commutative Fréchet algebra and fix an element $a \in \mathcal{A}$. We define the set of torsion elements (with respect to $a$) to be

$$\mathcal{T}(a) \equiv \{ v \in \mathcal{A}^\mathbb{N} \mid a^n v = 0 \text{ for some } n \in \mathbb{N} \}$$
It is routine to check that the torsion elements form an ideal of \( \mathcal{A}^\# \) (which is generally not closed).

**Definition 1.8.** Let \( \mathcal{A} \) be a commutative Fréchet algebra and fix an element \( a \in \mathcal{A} \). We define the *height* of an element \( v \) (with respect to \( a \)) to be

\[
h(v) = 0 \text{ if } v \notin a\mathcal{A}^\# ,
\]

\[
h(v) = n \text{ if } v \in a^n\mathcal{A}^\# - a^{n+1}\mathcal{A}^\# ,
\]

and

\[
h(v) = \infty \text{ if } v \in \bigcap_{n=1}^{\infty} a^n\mathcal{A}^\#
\]

We note that since \( a^{n+1}\mathcal{A}^\# \subseteq a^n\mathcal{A} \), the set \( \bigcap_{n=1}^{\infty} a^n\mathcal{A}^\# \) of elements of infinite height is also equal to \( \bigcap_{n=1}^{\infty} a^n\mathcal{A} \). It is routine to check that the set of elements of infinite height form an ideal but one important feature of this ideal is that it has a Fréchet topology \( \tau_h \) which is (formally) stronger than the original Fréchet topology \( \tau \) on \( \mathcal{A} \), namely the topology generated from the range space topology on each \( a^n\mathcal{A} \). A sequence \( \{v_i\}_{i=1}^{\infty} \) converges in \( \bigcap_{n=1}^{\infty} a^n\mathcal{A} \) if and only if for each \( n \in \mathbb{N} \) it is the image (under multiplication by \( a^n \)) of a convergent sequence in \( \mathcal{A} \).

There is a relationship between torsion elements and divisibility.

**Definition 1.9.** Let \( \mathcal{A} \) be a Fréchet algebra and fix an element \( a \in \mathcal{A} \). We say that a subspace \( H \) of \( \mathcal{A} \) is \( a \)-*divisible* if for every \( \lambda \in \mathbb{C} \) we have \( (\lambda - a)H = H \) (if \( \mathcal{A} \) is non-commutative this is usually qualified as "left-divisible"). There is always a largest \( a \)-divisible subspace which we will denote \( \mathcal{D}_a \) (which might be \( \{0\} \)).

**Lemma 1.10.** Let \( \mathcal{A} \) be a commutative Fréchet algebra and let \( a \in \mathcal{A} \) satisfy \( \mathcal{T}(a) \subseteq a\mathcal{A} \). Then

(i.) the ideal of torsion elements \( \mathcal{T}(a) \) is \( a \)-divisible, and

(ii.) \( a\bigcap_{n=1}^{\infty} a^n\mathcal{A} = \bigcap_{n=1}^{\infty} a^n\mathcal{A} \).

**Proof.** First note that \( \mathcal{T}(a) \) is closed under division by \( a \), that is, if \( aw = v \) then \( v \in \mathcal{T}(a) \) implies that \( w \in \mathcal{T}(a) \). This shows that \( a\mathcal{T}(a) = \mathcal{T}(a) \). Let \( \lambda \in \mathbb{C} \) be non-zero and let \( v \in \mathcal{T}(a) \). There exists \( n \in \mathbb{N} \) such that \( a^n v = 0 \). Choose polynomials \( p \) and \( q \) so that \( p(x)(\lambda - x) + q(x)x^n = 1 \). Let \( x = a \) and apply this polynomial equation to \( v \)

\[
v = p(a)(\lambda - a)v + q(a)a^n v = (\lambda - a)(p(a)v)
\]
This shows that \((\lambda - a)\mathcal{T}(a) = \mathcal{T}(a)\) for all \(\lambda \neq 0\), completing the proof of (i). Now let \(v \in (\cap_{n=1}^{\infty} a^n \mathcal{A})\). Hence there exists \(v_n \in \mathcal{A}\) such that \(a^n v_n = v\) for all \(n \in \mathbb{N}\). It suffices to show that \(v_1 \in (\cap_{n=1}^{\infty} a^n \mathcal{A})\). But, \((a^{n-1}v_n - v_1) \in \mathcal{T}(a)\). Since \(\mathcal{T}(a)\) is \(a\)-divisible, we can find \(w_n \in \mathcal{T}(a)\) such that \(a^{n-1}w_n = a^{n-1}v_n - v_1\) for all \(n \in \mathbb{N}\). This shows that \(v_1 \in (\cap_{n=1}^{\infty} a^{n-1} \mathcal{A}) = (\cap_{n=1}^{\infty} a^n \mathcal{A})\), ending the proof of (ii.).

We will have to handle one special case involving closed range when \(\mathcal{A}\) is a Banach algebra in the next section.

**Lemma 1.11.** Let \(\mathcal{A}\) be a commutative Banach algebra. Let \(a \in \mathcal{A}\) satisfy \(a\mathcal{A} = \overline{a\mathcal{A}}\) and \(\mathcal{T}(a) \subseteq a\mathcal{A}\). There exists a constant \(K\) such that whenever \(n \in \mathbb{N}\) and \(\{a^n e_i\}_{i=1}^{\infty}\) is a sequence satisfying \(\|a^n e_i - a^n e_{i+1}\| < 2^{-i}\) for all \(i \in \mathbb{N}\), we can find a new sequence \(\{a^{n-1} f_i\}_{i=1}^{\infty}\) satisfying

(i) \(a(a^{n-1} f_i) = a^n e_i\) for all \(i \in \mathbb{N}\).

(ii) \(\|a^{n-1} f_i - a^{n-1} f_{i+1}\| < K 2^{-i}\) for all \(i \in \mathbb{N}\).

**Proof.** Since the range of the multiplication operator \(M_a\) on \(\mathcal{A}\) is closed, the open mapping theorem guarantees the existence of a constant \(K\) such that whenever \(x = ay\) we can find \(z\) such that \(x = az\) with \(\|z\| \leq K\|x\|\) and \((y - z) \in \mathcal{N}_1 \equiv \{w \in \mathcal{A} \mid aw = 0\} \subseteq \mathcal{T}(a)\). We also have that \(a\mathcal{T}(a) = \mathcal{T}(a)\) by Lemma 1.10.

Start the induction by letting \(f_1 = e_1\) so that \(a(a^{n-1} f_1) = a^n e_1\) and \(\|a^n f_1 - a^n e_2\| < 2^{-1}\). By our above remarks we can find \(y \in \mathcal{A}\) such that \(ay = (a^n f_1 - a^n e_2)\) and \(\|y\| < K 2^{-1}\), with \(y = (a^{n-1} f_1 - a^{n-1} e_2) = h_2 \in \mathcal{N}_1 \subseteq \mathcal{T}(a)\). Since \(\mathcal{T}(a)\) is \(a\)-divisible we can find \(g_2 \in \mathcal{T}(a)\) such that \(a^{n-1} g_2 = h_2\) and, hence, \(a^n g_2 = 0\). Let \(f_2 = e_2 - g_2\) and compute

\[ a(a^{n-1} f_2) = a^n e_2 - a^n g_2 = a^n e_2, \]

\[ \|a^{n-1} f_1 - a^{n-1} f_2\| = \|a^{n-1} f_1 - a^{n-1} e_2 + a^{n-1} g_2\| = \|y\| < K 2^{-1}, \]

and

\[ \|a^{n} f_2 - a^n e_3\| = \|a(a^{n-1} e_2 - a^{n-1} g_2) - a^n e_3\| = \|a^n e_2 - a^n e_3\| < 2^{-2} \]

Continue replacing \(e_i\)’s by \(f_i\)’s until at the \(n\)-th stage we have

\[ a^n f_i = a^n e_i \text{ for } i = 1, 2, \ldots n, \]

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\[ \|a^{n-1} f_i - a^{n-1} f_{i+1}\| < K2^{-i} \text{ for } i = 1, 2, \ldots (n-1), \]

and

\[ \|a^n f_n - a^n e_{n+1}\| < 2^{-n} \]

Again we can find \( y \in \mathcal{A} \) such that \( ay = (a^n f_n - a^n e_{n+1}) \) and \( \|y\| < K2^{-n} \), with \( y - (a^{n-1} f_n - a^{n-1} e_{n+1}) = h_{n+1} \in \mathcal{N}_1 \subseteq \mathcal{T}(a) \). Since \( \mathcal{T}(a) \) is \( a \)-divisible we can find \( g_{n+1} \in \mathcal{T}(a) \) such that \( a^{n-1} g_{n+1} = h_{n+1} \) and, hence, \( a^n g_{n+1} = 0 \). Let \( f_{n+1} = e_{n+1} - g_{n+1} \) and compute

\[ a(a^{n-1} f_{n+1}) = a^n e_{n+1} - a^n g_{n+1} = a^n e_{n+1} , \]

\[ \|a^{n-1} f_n - a^{n-1} f_{n+1}\| = \|a^{n-1} f_n - a^{n-1} e_{n+1} + a^{n-1} g_{n+1}\| = \|y\| < K2^{-n} , \]

and

\[ \|a^n f_{n+1} - a^n e_{n+2}\| = \|a(a^{n-1} e_{n+1} - a^{n-1} g_{n+1}) - a^n e_{n+2}\| \]

\[ = \|a^n e_{n+1} - a^n e_{n+2}\| < 2^{-(n+1)} \]

Induction now establishes the theorem. \( \blacksquare \)

**Corollary 1.12.** Let \( \mathcal{A} \) be a commutative Banach algebra. Let \( a \in \mathcal{A} \) satisfy \( a\mathcal{A} = \overline{a\mathcal{A}} \) and \( \mathcal{T}(a) \subseteq a\mathcal{A} \). Then all ranges are closed (\( a^n \mathcal{A} = \overline{a^n \mathcal{A}} \) for all \( n \in \mathbb{N} \)), the ideal consisting of the elements of infinite height \( \cap_{n=1}^{\infty} a^n \mathcal{A} \) is closed, and \( \rho_\sigma(a) > 0 \).

**Proof.** Let \( n \in \mathbb{N} \) and \( y \in \overline{a^n \mathcal{A}} \). We can certainly find a sequence \( \{a^n e_i\}_{i=1}^{\infty} \) converging to \( y \) satisfying \( \|a^n e_i - a^n e_{i+1}\| < 2^{-i} \) for all \( i \in \mathbb{N} \). Applying Lemma 1.11 \( n \)-times gives us a sequence \( \{g_i\}_{i=1}^{\infty} \) satisfying \( a^n g_i = a^n e_i \) and \( \|g_i - g_{i+1}\| < K^{n}2^{-i} \) for all \( i \in \mathbb{N} \). Since \( \lim_{i \to \infty} g_i = g \) for some \( g \in \mathcal{A} \) we must have \( a^n g = y \). Therefore \( a^n \mathcal{A} = \overline{a^n \mathcal{A}} \). Since \( n \) was arbitrary this shows that all ranges are closed. In consequence

\[ \cap_{n=1}^{\infty} a^n \mathcal{A} = \cap_{n=1}^{\infty} \overline{a^n \mathcal{A}} = \overline{\cap_{n=1}^{\infty} a^n \mathcal{A}} \]

and consequently, Lemma 1.10 shows that

\[ a(\cap_{n=1}^{\infty} a^n \mathcal{A}) = \overline{\cap_{n=1}^{\infty} a^n \mathcal{A}} \]

If \( a \) were quasinilpotent we would also have

\[ (\lambda - a)(\cap_{n=1}^{\infty} a^n \mathcal{A}) = \overline{\cap_{n=1}^{\infty} a^n \mathcal{A}} \]

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for all $\lambda \neq 0$ and the ideal consisting of the elements of infinite height would be $a$-divisible. Since it contains the torsion elements $T(a)$ and since a quasinilpotent operator with closed range cannot be injective, the torsion subspace cannot be trivial (i.e. $T(a) \neq \{0\}$). But a closed divisible subspace in a Banach space must be trivial since if $\lambda$ is in the boundary of the spectrum $\sigma(a)$ of $a$ then $(\lambda - a)$ cannot be surjective. This contradiction shows that $a$ is not quasinilpotent and $\rho_\sigma(a) > 0$, finishing the proof of the corollary. 

We now have a number of related technical definitions which have been used before (see [Allan1] and [Allan2]) and will be important in the subsequent sections.

**Definition 1.13.** Let $a$ be an element in a Fréchet algebra $\mathcal{A}$.

(i.) We say that $a$ has **finite closed descent** if there exists $n \in \mathbb{N}$ such that $a^n \in a^{n+1}\mathcal{A}^\sharp$.

(ii.) We say that $a$ has **locally finite closed descent** if for every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $a^n \in a^{n+1}\mathcal{A}^\sharp_N$ (the closure is taken in the $N$-th seminorm). Note that if $\mathcal{A}$ is not a Banach space, $n$ in general will depend on $N$.

(iii.) We say that $a$ has a **non-trivial approximate identity** if there exists a sequence $\{e_i\}_{i=1}^\infty$ in $\mathcal{A}$ satisfying $\sigma(e_i) = \{0\}$ for all $i \in \mathbb{N}$ and $\lim_{i \to \infty} ae_i = a$.

We remark that if Definition 1.13(ii) holds, it is easily checked, as a consequence of the submultiplicative property of the seminorms, that

$$a^n\mathcal{A}^\sharp \subseteq a^{n+1}\mathcal{A}^\sharp N = a^{n+1}a^n\mathcal{A}^\sharp N \subseteq a^n\mathcal{A}^\sharp N$$

Taking closures of the above chain of containments with respect to the $N$-th seminorm shows that multiplication by $a$ on $a^n\mathcal{A}^\sharp N$ has $\| \cdot \|_N$-dense range. It is then a routine induction to show that for every $p \in \mathbb{N}$, multiplication by $a^p$ on $a^n\mathcal{A}^\sharp N$ has $\| \cdot \|_N$-dense range, so that

$$\overline{a^{n+p}\mathcal{A}^\sharp N} = a^n\mathcal{A}^\sharp N \quad \text{and} \quad \overline{a^{n+p}} \subseteq \overline{a^{n+p+1}}\mathcal{A}^\sharp N$$

For the next definition we stress that we are not assuming that $\mathcal{A}$ is an integral domain.
Definition 1.14. Let $\mathcal{A}$ be a commutative Fréchet algebra. We say that an element $a \in \mathcal{A}$ divides $b \in \mathcal{A}$ (and we write $a|b$) if there exists $c \in \mathcal{A}^\sharp$ (note the sharp) such that $b = ac$. We say that an element $a \in \mathcal{A}$ is prime if whenever $a|bc$ then either $a|b$ or $a|c$. We say that an element $a \in \mathcal{A}$ is semiprime if whenever $a|b^2$ then $a|b$.

It is a consequence of [Read1] that every semiprime element in a commutative Banach algebra has closed range. With regard to formal power series, it is too much to ask that the generator be prime (unless we are in the very special situation of the coefficient algebra being an integral domain). We will therefore work with the less stringent requirement of being weakly almost prime. The definition is unfortunately less direct than one might like. Although the two concepts can easily differ in the context of Fréchet algebras, it is an open question whether or not there exists a weakly almost prime element of a commutative radical Banach algebra which is not prime.

Definition 1.15. Let $q \in \mathcal{A}$, $k \in \mathbb{N}$ with $k \geq 2$ and $\{x_1, x_2, \ldots, x_k\} \subseteq \mathcal{A}$. We say that $\{x_1, x_2, \ldots, x_k\}$ satisfies subdivisibility with respect to $q$ if

\[ q|x_r x_s \text{ for } r < s \]
\[ q^2|x_r x_s x_t \text{ for } r < s < t \]
\[ \ldots \]
\[ q^{k-1}|x_1 x_2 \ldots x_k \]

We say that $q$ is almost $k$-prime if whenever $\{x_1, x_2, \ldots, x_k\}$ satisfies subdivisibility with respect to $q$ then there exists $\{u_1, u_2, \ldots, u_k\} \subseteq \mathcal{A}^\sharp$ such that

\[ \prod_{j=1}^{k} (x_j - qu_j) = 0 . \]

We say that $q$ is weakly almost $k$-prime if whenever $\{x_1, x_2, \ldots, x_k\}$ satisfies subdivisibility with respect to $q$ then there exists $\{u_1, u_2, \ldots, u_k\} \subseteq \mathcal{A}^\sharp$ such that

\[ \prod_{j=1}^{k} (x_j - qu_j) \in q^k \mathcal{A}^\sharp . \]

Definition 1.16. Let $p$ be an element of $\mathcal{A}$. We say that $p$ is almost prime if $p^k$ is almost $(k+1)$-prime for all $k \in \mathbb{N}$. We say that $p$ is weakly almost prime if $p^k$ is weakly almost $(k+1)$-prime for all $k \in \mathbb{N}$.
§2. Local Power Series Quotients. We are interested when a commutative Fréchet algebra $A^\mathbb{I}$ has a (unital) algebra homomorphism $\theta$ which is onto an algebra of formal series with coefficients in some commutative unital complex algebra $A_0$. We will use the notation

$$A_0[[X]] \equiv \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in A_0 \right\}$$

for such an algebra of formal series. If the kernel of $\theta$ is the ideal $J$ then we have that $\theta$ factors through $J$ as an isomorphism $\theta \cong$ from $A^\mathbb{I}/J$ onto $A_0[[X]]$. We do not, at this point, assume any topology on $A_0$. Unfortunately, if we put no restrictions on $J$ there may be some very ill behaved isomorphisms (see [Dales-McClure, Theorem 2.3]) even in the case where $A_0 \simeq \mathbb{C}$.

If there is a unital algebra homomorphism $\theta$ from $A^\mathbb{I}$ onto $A_0[[X]]$ with kernel of $\theta$ equal to $J$ and associated isomorphism $\theta \cong$ from $A^\mathbb{I}/J$ onto $A_0[[X]]$ some element $x$ must be mapped to the indeterminate $X$, and since $X$ is not invertible in $A_0[[X]]$, it must be the case that $x$ is not invertible in $A^\mathbb{I}$. If $\theta(x) = X$ then, considering the ideal consisting of the elements of infinite height (with respect to $x$), it is clear that $\cap_{i=0}^{\infty} x^n A$ must be contained in the kernel $J$ since $X$ has no elements of infinite height in $A_0[[X]]$. Since $\cap_{i=0}^{\infty} x^n A$ has a (formally) stronger Fréchet topology, it is therefore natural to investigate the special case where the kernel of $\theta$ is precisely the ideal consisting of the elements of infinite height (with respect to $x$). We call this a formal power series quotient based at $x$ and refer to $x$ as the generating element.

**Definition 2.1.** Let $x \in A$ and suppose that the ideal generated by $x$, namely $x A^\mathbb{I}$, is a non-zero, proper ideal of $A$ (this excludes trivial cases). We say that the commutative Fréchet algebra $A$ has a formal power series quotient based at $x$ if there exists a commutative unital complex algebra $A_0$ and a unital algebra homomorphism $\theta$ onto $A_0[[X]]$ such that the kernel of $\theta$ is precisely the ideal consisting of the elements of infinite height, that is

$$\theta \cong : (A^\mathbb{I}/ \cap_{n=1}^{\infty} x^n A) \cong A_0[[X]]$$

and $\theta(x) = X$. We say that a commutative Fréchet algebra $A$ has a local power series quotient if it has a formal power series quotient based at $x$ for some $x \in A$ for which the ideal generated by $x$ is non-zero and proper.
We remark that Definition 2.1 generalizes to non-commutative Fréchet algebras by replacing the kernel of \( \theta \) above, \( \cap_{n=1}^{\infty} x^n \mathcal{A} \), by \( \cap_{n=1}^{\infty} (\mathcal{C} + x^n \mathcal{A}) \) where \( \mathcal{C} \) is the ideal generated by the commutators \( \{ xy - yx \mid x, y \in \mathcal{A} \} \). However, this has the unfortunate consequence that the kernel of \( \theta \) is no longer an intersection of range spaces. We will have some additional remarks to make concerning the non-commutative situation in Section 3.

It is useful to consider our previous example (1.3) \( \mathcal{A} = C^\infty(\mathbb{R}) \), which is already unital. Let \( x \) be the function \( x(t) = t \) for all \( t \in \mathbb{R} \) and let \( f \in C^\infty(\mathbb{R}) \). If we define \( \theta(f) \) to simply be the Taylor series of \( f \) expanded about \( t = 0 \) we certainly have a homomorphism from \( C^\infty(\mathbb{R}) \) onto \( \mathcal{C}[[X]] \) which takes the function \( x \) to \( X \) (here \( \mathcal{A}_0 = \mathcal{C} \)). If \( f \in \cap_{n=1}^{\infty} x^n C^\infty(\mathbb{R}) \) then certainly \( \theta(f) = 0 \). If \( \theta(f) = 0 \) then, for each \( n \in \mathbb{N} \), the \( n \)-th derivative of \( f \) at zero satisfies \( f^{(n)}(0) = 0 \). But if \( f \in C^\infty(\mathbb{R}) \) has a zero of infinite order at \( t = 0 \) it must be divisible by arbitrarily high powers of \( x \).

This shows that the ideal consisting of the elements of infinite height is precisely the set of those \( C^\infty(\mathbb{R}) \) functions which vanish at \( 0 \) and whose derivatives all vanish at \( 0 \). We then have

\[
\theta_{\mathcal{M}} : \left( C^\infty(\mathbb{R}) / \cap_{n=1}^{\infty} x^n C^\infty(\mathbb{R}) \right) \cong \mathcal{C}[[X]]
\]

with \( \mathcal{A}_0 = \mathcal{C} \), and \( C^\infty(\mathbb{R}) \) has a local power series quotient.

We require several lemmas. The first lemma, a somewhat more general Mittag-Leffler theorem (see [Sinclair]) is proved by essentially the same method as [Thomas, Lemma 1.1d] although the maps here are not assumed to commute.

**Lemma 2.2.** Let \( \{ X_n \}_{n=1}^{\infty} \) be closed subspaces of a Frechet space \( (X, \| \cdot \|_n) \) (with \( \| \cdot \|_n \leq \| \cdot \|_{n+1} \)) and let \( \{ \varphi_n : X_{n+1} \to X_n \}_{n=1}^{\infty} \) be a sequence of continuous functions satisfying

(i.) There exists a sequence of constants \( \{ c_i \}_{i=1}^{\infty} \) such that for all \( a, b \in X_{n+1} \) and non-negative integers \( p \) we have

\[
\| \varphi_n(a) - \varphi_n(b) \|_{n+p} \leq c_n \| a - b \|_{n+p}
\]

for \( n = 1, 2, \ldots \).

(ii.) \( \varphi_n(X_{n+1}) \supseteq X_n \) for \( n = 1, 2, \ldots \).
Then the inverse, or projective, limit \( P = \{(y_n)_{n=1}^{\infty} \mid \varphi_n y_{n+1} = y_n \text{ for } n = 1, 2, \ldots \} \) is non-empty and \( \overline{\pi_n(P)}\supseteq X_n \) (where \( \pi_n \) is the \( n \)-th coordinate projection) for \( n = 1, 2, \ldots \).

**Proof.** Without loss of generality we may assume that \( C_i \geq 1 \) for all \( i \in \mathbb{N} \). Fix \( n \in \mathbb{N} \) and \( \epsilon > 0 \). Let \( x_n \) be any element of \( X_n \). Choose \( x_{n+1} \in X_{n+1} \) so that \( \|\varphi_n x_{n+1} - x_n\|_n < \epsilon/(2^n C_n) \). Continue inductively choosing \( x_{n+p+1} \in X_{n+p+1} \) so that

\[
\|\varphi_n x_{n+p+1} - x_n\|_{n+p} < \epsilon/(2^{n+p+1} C_{n+p}^p)
\]

for \( p = 1, 2, \ldots \). Given any non-negative integer \( j \), observe that

\[
\|\varphi_{n+j} x_{n+j+1} - x_{n+j}\|_n + \sum_{p=j+1}^{\infty} \|\varphi_{n+j} \varphi_{n+j+1} \ldots \varphi_{n+p} x_{n+p+1} - \varphi_{n+j} \varphi_{n+j+1} \ldots \varphi_{n+p-1} x_{n+p}\|_{n+p}
\]

\[
\leq \|\varphi_{n+j} x_{n+j+1} - x_{n+j}\|_n + \sum_{p=j+1}^{\infty} C_{n+p} \|\varphi_{n+j} \varphi_{n+j+1} \ldots \varphi_{n+p} x_{n+p+1} - \varphi_{n+j+1} \ldots \varphi_{n+p-1} x_{n+p}\|_{n+p}
\]

\[
\cdots
\]

\[
\leq \sum_{p=j}^{\infty} C_{n+p}^{p-j} \|\varphi_{n+p} x_{n+p+1} - x_{n+p}\|_{n+p}
\]

\[
\leq \sum_{p=j}^{\infty} C_{n+p}^{p-j} \epsilon/(2^{n+p+1} C_{n+p}^p) \leq \epsilon/(2^{n+j})
\]

This shows that the sequence \( \{\varphi_{n+j} \ldots \varphi_{n+p} x_{n+p+1}\}_{p=j}^{\infty} \) is Cauchy (in \( p \)) in the Fréchet topology of \( X \) (and \( X_{n+j} \), which is closed). Consequently, there exists \( y_{n+j} \in X_{n+j} \) such that

\[
\varphi_{n+j} \ldots \varphi_{n+p} x_{n+p+1} \rightarrow y_{n+j} \text{ as } p \rightarrow \infty
\]

for \( j = 0, 1, 2, \ldots \). But if we also define \( y_{n-1} = \varphi_{n-1} y_n \), \( y_{n-2} = \varphi_{n-2} \varphi_{n-1} y_n \), \ldots \( y_1 = \varphi_1 \varphi_2 \ldots \varphi_{n-1} y_n \), then it is routine to show that \( (y_i)_{i=1}^{\infty} \in P \) and \( \pi_n (y_i)_{i=1}^{\infty} = y_n \).

Finally we can compute the \( n \)-th seminorm distance from \( \pi_n (y_i)_{i=1}^{\infty} \) and \( x_n \) as follows:

\[
\|y_n - x_n\|_n \leq \lim_{m \rightarrow \infty} \|\varphi_n \ldots \varphi_{n+m} x_{n+m+1} - x_n\|_n
\]
\[
\leq \lim_{m \to \infty} \left\| \sum_{p=0}^{m} \varphi_n \cdots \varphi_{n+p} x_{n+p+1} - \varphi_n \cdots \varphi_{n+p-1} x_{n+p} \right\|_{n+p}
\]
\[
\leq \sum_{p=0}^{\infty} \left\| \varphi_n \cdots \varphi_{n+p} x_{n+p+1} - \varphi_n \cdots \varphi_{n+p-1} x_{n+p} \right\|_{n+p}
\]
\[
\leq \sum_{p=0}^{\infty} \frac{\epsilon}{(2^{n+p+1})} \leq \epsilon
\]

Since \( \epsilon > 0 \) was arbitrary this shows that the closure in the \( n \)-th seminorm \( \overline{\pi_n(P)^n} \) contains \( X_n \). \( \blacksquare \)

**Lemma 2.3.** Let \( x \) be an element of a commutative Fréchet algebra \( \mathcal{A} \), let \( \hat{x} \) denote the coset containing \( x \) in the quotient Fréchet algebra \( \mathcal{A}^2 / \bigcap_{m=1}^{\infty} x^m \mathcal{A} \). Then *either* there exists \( n \in \mathbb{N} \) such that

\[
x^k \not\in x^{k+1} \mathcal{A}^m
\]

for all \( k \in \mathbb{N} \) (i.e. \( x \) does not have locally finite closed descent) *or* \( \hat{x} \) is locally nilpotent.

**Proof.** Suppose that the first alternative fails and that no such \( n \) exists. Then for any \( n \in \mathbb{N} \) we can find \( k \in \mathbb{N} \) such that

\[
x^k \in \overline{x^{k+1} \mathcal{A}^m} = x(\overline{x^k \mathcal{A}^m})
\]

Our remark after Definition 1.13 shows that

\[
\overline{x^{k+p} \mathcal{A}^m} = x^p \overline{(x^k \mathcal{A}^m)} = x(x^{k+p-1} \mathcal{A}^m) = x(x^k \mathcal{A}^m) = x^k \mathcal{A}^m
\]

for \( p = 1, 2, \ldots \). It is therefore possible to choose an increasing sequence \( \{k_n\}_{n=1}^{\infty} \) such that

\[
x(x^{kn+1} \mathcal{A}^m) = x^{kn} \mathcal{A}^m
\]

for \( n = 1, 2, \ldots \). Therefore, for each \( n \in \mathbb{N} \), let

\[
X_n = \overline{x^{kn} \mathcal{A}^m},
\]

let \( \varphi_n \) be multiplication by \( x \) (which is a continuous function from \( X_{n+1} \) to \( X_n \)) and let \( C_n = \|x\|_n \). Note that the hypotheses of Lemma 2.2 are satisfied, namely
(i.) For all \( n, p \in \mathbb{N} \) the difference in the \((n+p)\)-th seminorms has the correct upper bound:

\[
\| \varphi_n(a) - \varphi_n(b) \|_{n+p} = \| x(a-b) \|_{n+p} \leq \| x \|_{n+p} \| a-b \|_{n+p} = C_{n+p} \| a-b \|_{n+p}
\]

(ii.) For all \( n \in \mathbb{N} \), the maps have dense range:

\[
\varphi_n(X_{n+1})^n = x(x(k_{n+1}A)^{n+1})^n = x(k_{n+1}A^n)^n = x(k_{n}A^n) = X_n
\]

Lemma 2.2 can therefore be applied. In this case, it is easily seen that the image of any canonical projection of the projective limit is simply the largest subspace \( H \subseteq A^\ast \) satisfying \( xH = H \) (i.e. the subspace is divisible by the single operation of multiplication by \( x \)), so that

\[
\overline{H}^n \supseteq x(k_{n}A^n) \supseteq x(k_{n}A^\ast)
\]

for \( n = 1, 2, \ldots \). Since we must have \( H \subseteq (\cap_{m=1}^\infty x^mA) \) it follows that

\[
x(k_{n}) \subseteq x(k_{n}A^\ast) \subseteq \overline{H}^n \subseteq \cap_{m=1}^\infty x^mA^n
\]

for \( n = 1, 2, \ldots \). This shows that \( x \) is locally nilpotent and completes the proof. \( \blacksquare \)

Next we observe that the inductive construction (of what we called a recalcitrant system) in [Thomas2] of sections 3.10a–3.10d can be reworded in our setting to equivalently read:

**Lemma 2.4.** [Thomas2] Let \( s \) be a non-nilpotent element of a commutative radical Banach algebra \( \mathcal{R}^\ast \) with identity adjoined (so that \( \mathcal{R}^\ast \cong \mathfrak{F} \cdot 1 \oplus \mathcal{R} \) and \( \mathcal{R} = \text{rad}(\mathcal{R}^\ast) \)). Let \( \{e_\alpha\}_{\alpha \in \mathbb{N}} \) be a sequence of elements in \( \mathcal{R} \) such that \( \lim_{\alpha \to \infty} e_\alpha s = s \). Fix a positive integer \( k \in \mathbb{N} \) and choose \( k \) fixed sequences \( \{\lambda_{ij}\}_{i,j=1}^\infty \) satisfying

\[
\lim_{j \to \infty} |\lambda_{i1} \lambda_{i2} \ldots \lambda_{ij}| = +\infty
\]

for \( i = 1, 2, \ldots, k \). Order the set of double indices \( \{\{ij\}_{i,j=1}^\infty\}_{i,j=1}^\infty \) with the following lexicographical order: \( i_1j_1 < i_2j_2 \) if either \( j_1 < j_2 \) or \( j_1 = j_2 \) and \( i_1 < i_2 \). Once \( \alpha_{ij} \) has been chosen the element \((\lambda_{ij} + (1 - \lambda_{ij})e_{\alpha_{ij}})\) will be invertible in \( \mathcal{R}^\ast \). Then, as long as the values \( \{\alpha_{ij}\}_{i,j=1}^\infty \) are chosen to increase sufficiently rapidly as the \( \{ij\}_{i=1}^\infty \) increase in the lexicographical order, the set of elements

\[
s_i = s \prod_{j=1}^\infty (\lambda_{ij} + (1 - \lambda_{ij})e_{\alpha_{ij}})
\]

for \( i = 1, 2, \ldots, k \). The resulting \( s \) is a non-nilpotent element of \( \mathcal{R}^\ast \).
converge in $\mathcal{R}$ for $i = 1, 2, \ldots, k$, satisfy subdivisibility with respect to $s$, and have the following property:

$$\sup_n \left\| \prod_{i=1}^k (s_i - sv_i)^n \right\|^{1/n} = +\infty$$

for all choices of $\{v_1, v_2, \ldots, v_k\} \subseteq \mathcal{R}^d$.

We note that the requirement in the above lemma that the $\{e_\alpha\}_{\alpha \in \mathbb{N}}$ be in the radical $\mathcal{R}$ (and not simply in $\mathcal{R}^d$) is essential for the contraction given in [Thomas2]. We wish to extend this result to radical Fréchet algebras but the more stringent requirement that the element be non-locally nilpotent is necessary.

**Corollary 2.5.** Let $\mathcal{R}^d$ be a commutative radical Fréchet algebra with identity adjoined (so that $\mathcal{R}^d \cong \mathbb{C} \cdot 1 \oplus \mathcal{R}$ and $\mathcal{R} = \text{rad}(\mathcal{R}^d)$) and let $x$ be a non-locally nilpotent element of $\mathcal{R}$ satisfying

$$x^{m_0} \in \overline{x^{m_0} \mathcal{R}}$$

for some $m_0 \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$, and elements $\{s_1, s_2, \ldots, s_{m_0+1}\} \subseteq \mathcal{R}$ such that $\{s_1, s_2, \ldots, s_{m_0+1}\}$ satisfy subdivisibility with respect to $x^{m_0}$ and

$$\sup_n \left\| \prod_{i=1}^{m_0+1} (s_i - x^{m_0}u_i)^n \right\|^{1/n} = +\infty$$

for all choices of $\{u_1, u_2, \ldots, u_{m_0+1}\} \subseteq \mathcal{R}^d$. In consequence, $x$ is *not* weakly almost prime.

**Proof.** Since $x$ is non-locally nilpotent we can choose $n_0 \in \mathbb{N}$ such that $\|x^n\|_{n_0} \neq 0$ for all $n \in \mathbb{N}$. Let $k = m_0+1$ and choose $k$ fixed sequences $\{\{\lambda_{ij}\}_{i=1}^{k}\}_{j=1}^{\infty}$ satisfying

$$\lim_{j \to \infty} |\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_j}| = +\infty$$

for $i = 1, 2, \ldots, k$. We have, as a consequence of the Arens-Michael isomorphism, a continuous homomorphism $B_{n_0}$ (with dense range) from $\mathcal{R}^d$ into $B_{n_0}$ with kernel $\mathcal{I}_{n_0}$. It is easy to verify that $B_{n_0}$ is also a radical algebra with identity adjoined since for all $r \in \mathcal{R}$

$$\{0\} = \sigma(r) = \bigcup_{n=1}^{\infty} \sigma_n(r)$$

as noted in the discussion of the Arens-Michael isomorphism in Section 1. But the spectrum is non-empty in each $B_n$ so $\sigma_{n_0}(r) = \sigma(\pi_{n_0}(r)) = \{0\}$ for all $r \in \mathcal{R}$.
\( \mathcal{R} \). This, together with the fact that \( \pi_{n_0} \) has dense range, shows that \( \mathcal{B}_{n_0} \) is a commutative radical Banach algebra with identity adjoined. Let \( s = (\pi_{n_0}(x))^{m_0} \). Since \( x^{m_0} \in x^{m_0} \mathcal{R} \) we can find a sequence \( \{f_{\alpha}\}_{\alpha \in \mathbb{N}} \) in \( \mathcal{R} \) such that \( f_{\alpha}x^{m_0} \rightarrow x^{m_0} \) in the Fréchet topology of \( \mathcal{R} \). If we let \( e_{\alpha} = \pi_{n_0}(f_{\alpha}) \) for each \( \alpha \in \mathbb{N} \) it will follow that \( \lim_{\alpha \to \infty} e_{\alpha}s = s \) in the norm topology of the commutative radical Banach algebra \( \mathcal{B}_{n_0} \) with identity adjoined. If \( \{\alpha_{ij}\}_{i,j=1}^{k} \) are chosen to increase sufficiently rapidly as the \( \{ij\}_{i,j=1}^{k} \) increase in the lexicographical order then the elements

\[
 s_i \equiv x^{m_0} \prod_{j=1}^{\infty} (\lambda_{ij} + (1 - \lambda_{ij})f_{\alpha_{ij}})
\]

will converge in the Fréchet topology of \( \mathcal{R} \) and satisfy subdivisibility with respect to \( x^{m_0} \), the conditions of Lemma 2.4 will be met, and the elements

\[
 \pi_{n_0}(s_i) \equiv s \prod_{j=1}^{\infty} (\lambda_{ij} + (1 - \lambda_{ij})e_{\alpha_{ij}})
\]

will converge in the Banach algebra topology of \( \mathcal{B}_{n_0} \), and

\[
 \sup_{n} \frac{\|\prod_{i=1}^{k} (\pi_{n_0}(s_i) - sv_i)^n\|^{1/n}}{\|sv^n\|^{1/n}} = +\infty
\]

for all choices of \( \{v_1, v_2, \ldots, v_k\} \subseteq \mathcal{B}_{n_0} \). Remembering that for \( u \in \mathcal{R}^d \), \( \|u\|_{n_0} = \|\pi_{n_0}(u)\| \), we conclude that given any \( k = m_0 + 1 \) choices \( \{u_1, u_2, \ldots, u_k\} \subseteq \mathcal{R}^d \), we can apply the above to \( v_i = \pi_{n_0}(u_i) \) for \( i = 1, 2, \ldots, k \) and obtain

\[
 \sup_{n} \frac{\|\prod_{i=1}^{m_0+1} (s_i - x^{m_0}u_i)^n\|^{1/n}}{\|x^{m_0(m_0+1)}\|^{1/n}} = +\infty
\]

Suppose now that \( x \) were weakly almost prime. Then \( x^{m_0} \) would be weakly almost \( (m_0 + 1) \)-prime. However, the set \( \{s_1, s_2, \ldots, s_{m_0+1}\} \) satisfies subdivisibility with respect to \( x^{m_0} \) and there would have to exist \( \{u_1, u_2, \ldots, u_{m_0+1}\} \subseteq \mathcal{R}^d \) and \( u \in \mathcal{R}^d \) such that

\[
 \prod_{i=1}^{m_0+1} (s_i - x^{m_0}u_i) = x^{m_0(m_0+1)}u
\]

But \( \sup_{n} \|u^n\|^{1/n}_{n_0} = \rho_{\sigma}(\pi_{n_0}(u)) \) is bounded in the Banach algebra \( \mathcal{B}_{n_0} \) contradicting

\[
 \sup_{n} \frac{\|\prod_{i=1}^{m_0+1} (s_i - x^{m_0}u_i)^n\|^{1/n}}{\|x^{m_0(m_0+1)}\|^{1/n}} = +\infty
\]
and showing that such a $u \in \mathcal{R}^d$ cannot exist. Therefore, $x$ cannot be weakly almost prime. \hfill \blacksquare

We now have our result which establishes the main necessary conditions for having a local power series quotient. We note that the seventh conclusion (vii.) is the most important since it shows that either $x$ has finite closed descent and $\mathcal{A}^d$ has non-trivial representation theory, or $x$ does not have finite closed descent, $\mathcal{A}^d$ is not a Banach algebra, and $\dot{x}$ must be locally nilpotent (but non-nilpotent) in the quotient Fréchet algebra $\mathcal{A}^d/\cap_{m=1}^{\infty} x^n \mathcal{A}.$

**Proposition 2.6.** Suppose that the commutative Fréchet algebra $\mathcal{A}$ has a formal power series quotient based at $x$, let $\theta$ denote the surjective homomorphism for which $\theta(x) = X$, let $\theta_{\cong}$ denote the associated isomorphism

$$\theta_{\cong} : \left( \mathcal{A}^d/\cap_{n=1}^{\infty} x^n \mathcal{A} \right) \cong \mathcal{A}_0[[X]],$$

and let $\dot{x}$ denote the coset containing $x$ in the quotient Fréchet algebra $\mathcal{A}^d/\cap_{m=1}^{\infty} x^m \mathcal{A}$. Then the following conditions hold:

(i.) $x$ is not nilpotent.

(ii.) The torsion elements $\mathcal{T}(x)$ with respect to $x$ are contained in $\cap_{n=1}^{\infty} x^n \mathcal{A}$ and hence $x(\cap_{n=1}^{\infty} x^n \mathcal{A}) = (\cap_{n=1}^{\infty} x^n \mathcal{A})$.

(iii.) (algebraically) $\mathcal{A}_0 \cong \mathcal{A}^d/x\mathcal{A}^d$.

(iv.) If $\mathcal{A}$ contains an idempotent $e$ different from the identity, and if $e$ is not contained in the kernel $(\cap_{n=1}^{\infty} x^n \mathcal{A})$ of $\theta$ then $b_0 = \theta(e)$ is a non-zero idempotent in $\mathcal{A}_0$ and $ex \neq 0$. Also, $e\mathcal{A}$ is a closed subalgebra of $\mathcal{A}$ and if we define $\theta_{\cong}(a) = \theta(ea)$ we have the following associated isomorphism

$$\theta_{\cong} : \left( e\mathcal{A}/\cap_{n=1}^{\infty} (ex)^n e\mathcal{A} \right) \cong (b_0 \mathcal{A}_0)[[X]]$$

so that $e\mathcal{A}$ has a formal power series quotient based at $(ex)$.

(v.) Either $\mathcal{A}_0$ contains non-zero idempotents other than the identity element or whenever $x = uv$ for $u, v \in \mathcal{A}^d$ one of $\{u, v\}$ is invertible modulo the kernel $(\cap_{n=1}^{\infty} x^n \mathcal{A})$ of $\theta$ and, consequently, $x$ is irreducible.

(vi.) $y$ is divisible by $x^n$ if and only if $\theta(y) = \sum_{k=0}^{\infty} y_k X^k$ satisfies $y_0 = y_1 = \ldots y_{n-1} = 0$, and, consequently, $x$ is weakly almost prime.
(vii.) either $x$ has finite closed descent and $\mathcal{A}^\sharp$ is not a commutative radical Fréchet algebra with identity adjoined or $\mathcal{A}$ is not a Banach algebra, $x$ does not have finite closed descent, $\sigma(\hat{x}) = \{0\}$, and $\hat{x}$ is locally nilpotent (but non-nilpotent).

**Proof.** Since $\theta(x) = X$ and since $X$ is non-nilpotent $x$ is non-nilpotent. This proves (i.).

Since $\mathcal{A}_0[[X]]$ has no $X$-torsion it must be the case that $\mathcal{T}(x)$ is contained in the kernel of $\theta$ which is precisely $\cap_{n=1}^\infty x^n\mathcal{A}$. This means that $\mathcal{T}(x) \subseteq \cap_{n=1}^\infty x^n\mathcal{A} \subseteq x\mathcal{A}$ and an application of Lemma 1.10 establishes (ii.).

Note that $\mathcal{A}_0 \cong \mathcal{A}_0[[X]]/(X\mathcal{A}_0[[X]])$. Consideration of the preimages shows that

$$\mathcal{A}^\sharp/x\mathcal{A}^\sharp \cong (\mathcal{A}^\sharp / \cap_{n=1}^\infty x^n\mathcal{A})/(x, \mathcal{A}^\sharp / \cap_{n=1}^\infty x^n\mathcal{A}) \cong \mathcal{A}_0[[X]]/(X\mathcal{A}_0[[X]])$$

and establishes (iii.).

Suppose that $e$ is an idempotent in $\mathcal{A}$ different from the identity. If $e \not\in \cap_{n=1}^\infty x^n\mathcal{A}$ then $\theta(e) = b_0 + \sum_{i=1}^{\infty} b_iX^i$ is a non-zero idempotent in $\mathcal{A}_0[[X]]$. It is clear that $b_0^2 = b_0$. We claim that $b_i = 0$ for all $i \geq 1$. First note that $b_1 = 2b_0b_1$ since $\theta(e)^2 = \theta(e)$. But, multiplying both sides by $b_0$ yields $b_0b_1 = 2b_0b_1$ which forces $b_0b_1 = 0$ and, in consequence, $b_1 = 0$. Suppose that we have shown that $b_1 = b_2 = \ldots b_{n-1} = 0$. Then $\theta(e) = b_0 + \sum_{i=n}^{\infty} b_iX^i$ and idempotence again requires that $b_n = 2b_0b_n$. Multiplication by $b_0$ again shows that $b_0b_n = 2b_0b_n$ and $b_n = 0$. Induction now shows that $\theta(e) = b_0 \neq 0$ since we are assuming that $e$ is not in the kernel of $\theta$.

This also shows that $ex \neq 0$ since $\theta(ex) = b_0X \neq 0$. Hence, $b_0\mathcal{A}_0$ is a commutative unital complex algebra with identity $b_0$. Correspondingly, $e\mathcal{A}$ is a subalgebra of $\mathcal{A}$ with identity $e$. It is closed since it is the null space of the multiplication operator $M_{1-e}$ on $\mathcal{A}$ where $M_{1-e}(a) = (a - ea)$. Define $\theta_e$ on $e\mathcal{A}$ by $\theta_e(ea) = b_0\theta(a)$, which is just the restriction of $\theta$ to $e\mathcal{A}$. This map is clearly onto $b_0\mathcal{A}_0[[X]]$. If $ea$ is in the kernel of $\theta_e$ this means that $ea \in x^n\mathcal{A}$ for all $n \in \mathbb{N}$. Multiplying by $e$ we have that $ea \in ex^n\mathcal{A} = (ex)^n e\mathcal{A}$ for all $n \in \mathbb{N}$. This establishes the surjective homomorphism $\theta$ and associated isomorphism $\theta_{e\infty}$. Suppose that $ex\mathcal{A} = e\mathcal{A}$. Apply this $n$-times to obtain that $x^n(e\mathcal{A}) = e\mathcal{A}$ for all $n \in \mathbb{N}$. But then $e = e^2 \in e\mathcal{A} \in \cap_{n=1}^\infty x^n\mathcal{A}$, a contradiction to the fact that $e$ is not in the kernel of $\theta$. We remark that this is where we need $e \in \mathcal{A}$ rather than the weaker assumption $e \in \mathcal{A}^\sharp$. Therefore, $e\mathcal{A}$ has a formal power series quotient based at $(ex)$. This establishes (iv.)
Suppose that $x = uv$ for some $u, v \in \mathbb{A}^\sharp$. Apply $\theta$ so that we have $X = \theta(u)\theta(v)$ and let the expansions be $\theta(u) = \sum_{i=0}^{\infty} a_i X^i$ and $\theta(v) = \sum_{i=0}^{\infty} b_i X^i$. Computation shows that $a_0 b_0 = 0$ and $a_0 b_1 + a_1 b_0 = 1$. Consequently, one of $\{a_0 b_1, a_1 b_0\}$ must be non-zero. Without loss of generality assume that $a_0 b_1 \neq 0$. Multiply the equation $a_0 b_1 + a_1 b_0 = 1$ by $a_0 b_1$ and use commutativity and the fact that $a_0 b_0 = 0$ to conclude that

$$(a_0 b_1)^2 = a_0^2 b_1^2 = a_0 b_1$$

So $e_0 = a_0 b_1$ is a non-zero idempotent, and clearly $X$ factors as

$$(e_0 + (1 - e_0)X)((1 - e_0) + e_0 X)$$

However, if the only non-zero idempotent of $\mathcal{A}_0$ is $1$ we conclude $a_0 b_1 = 1$ so that

$$b_1 \theta(u) = 1 - \sum_{i=1}^{\infty} (-b_1 a_i) X^i$$

If $\lambda \neq 0$ then any element of the form $\lambda - \sum_{i=1}^{\infty} c_i X^i$ is invertible in $\mathcal{A}_0[[X]]$ as follows

$$(\lambda - Y)^{-1} = \lambda^{-1}(1 - (\lambda^{-1} Y))^{-1} = \sum_{i=0}^{\infty} \lambda^{-(i+1)} Y^i$$

for any $Y \in X.\mathcal{A}_0[[X]]$. The series converges since each $i$-th power term of the result only depends on finitely many terms of the sum. Hence, there exists $y \in \mathbb{A}^\sharp$ such that $uy \in (1 + \cap_{n=1}^{\infty} x^n.\mathcal{A})$, so that $u$ is invertible modulo the kernel of $\theta$. In addition, multiplying the above relation by $v$ shows that $xy = uvy \in (v + \cap_{n=1}^{\infty} x^n.\mathcal{A})$, so there exists $a \in \mathcal{A}$ such that $xy = v + xa$ or $v = x(y - a)$ and, hence, $x|v$. This shows that $x$ is irreducible, and completes the proof of assertion (v.).

Suppose that $y \in \mathcal{A}$ and $y = x^n z$ for some $n \in \mathbb{N}$ and $z \in \mathbb{A}^\sharp$. Apply the homomorphism $\theta$ and suppose that $\theta(y) = \sum_{k=1}^{\infty} y_k X^k$ with $y_k \in \mathcal{A}_0$ for $k = 1, 2, \ldots$ and $\theta(z) = \sum_{k=1}^{\infty} z_k X^k$ with $z_k \in \mathcal{A}_0$ for $k = 1, 2, \ldots$ Since $y = x^n z$ it must be the case that $\sum_{k=1}^{\infty} y_k X^k = \sum_{k=n}^{\infty} z_k X^k$ in $\mathcal{A}_0[[X]]$. This can only happen if $y_0 = y_1 = y_2 = \ldots = y_{n-1} = 0$. Conversely, if the series for $\theta(y)$ is of the form $\sum_{k=n}^{\infty} y_k X^k$ it is clear that $y \in x^n.\mathbb{A}^\sharp$ since $\theta$ is onto and its kernel is precisely $\cap_{m=1}^{\infty} x^m.\mathcal{A}$ which contains $x^n.\mathbb{A}^\sharp$. Let $m_0$ be any positive integer. We need to show that $x^{m_0}$ is weakly almost $(m_0 + 1)$-prime. But the demonstration of this fact can be done in exactly the same manner as the proof of [Thomas2, Theorem 2.25] since we now know that an element $y$ is divisible by $x^n$ if and only if the first $n$ terms of its series $\theta(y)$ vanish. Since $m_0$ was arbitrary, this shows that $x$ is weakly almost prime, finishing the proof of assertion (vi.).
We finally consider assertion (vii.). We have already noted that \((\lambda - X)\) is invertible in \(A_0[[X]]\) if \(\lambda \neq 0\). This shows that for every \(\lambda \neq 0\) there exists \(u_\lambda\) in \(A^d\) such that

\[(\lambda - x)u_\lambda \in 1 + \bigcap_{n=1}^\infty x^n A\]

so certainly

\[(\lambda - x)u_\lambda \in 1 + \bigcap_{n=1}^\infty x^n A\]

Let a “dot” denote the coset in the Fréchet algebra \(A^d/\bigcap_{n=1}^\infty x^n A\). It follows that 

\[(\lambda - \dot{x})u_\lambda = \dot{1} \text{ for every } \lambda \neq 0, \text{ so that } \sigma(\dot{x}) = \{0\}.\]

Suppose that for some seminorm \(\| \cdot \|_n\) we have

\[x^k \notin \overline{x^{k+1} A^i_n}\]

for every \(k \in \mathbb{N}\). Let \(\pi_n\) be the canonical projection of the Arens-Michael isomorphism, and let \(B_n\) be the corresponding \(n\)-th Banach algebra (on which the \(\| \cdot \|_n\) seminorm is a norm). Since \(\pi_n\) has dense range, it must also be the case that

\[(\pi_n(x))^k \notin \overline{(\pi_n(x))^{k+1} B_n}\]

for every \(k \in \mathbb{N}\) (the closure being taken in the Banach algebra \(B_n\)). For each \(k \in \mathbb{N}\) we can choose a continuous linear functional \(\varphi_k\) in the dual of \(B_n\) with \(\varphi_k((\pi_n(x))^k) = 1\) but

\[\varphi_k((\pi_n(x))^{k+1} B_n) = \{0\}\]

For each \(k \in \mathbb{N}\) we can also choose a scalar \(\lambda_k\) in the complex field sufficiently large so that

\[|\lambda_k| > \left( \sum_{i=1}^{k-1} |\lambda_i| \|\varphi_k\| (\pi_n(x))^i \right) + k \|\varphi_k\|\]

Since the homomorphism \(\theta\) is onto we can find an element \(a \in A^d\) satisfying \(\theta(a) = \sum_{i=1}^\infty \lambda_i X^i\) and for each \(k \in \mathbb{N}\) there must also exist \(b_k \in A^d\) such that \(\theta(b_k) = \sum_{i=k+1}^\infty \lambda_i X^i\). Since \(b_k\) is divisible by \(x^{k+1}\) modulo the kernel of \(\theta\) we have that

\[b_k \in x^{k+1} A^d + (\bigcap_{m=1}^\infty x^m A) \subseteq \overline{x^{k+1} A^i_n}\]

This forces \(\varphi_k(\pi_n(b_k)) = 0\). Since

\[a \in \lambda_k x^k + \sum_{i=0}^{k-1} \lambda_i x^i + b_k + (\bigcap_{m=1}^\infty x^m A)\]

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we have that
\[ \| \varphi_k \|_n \| a \|_n \geq |\varphi_k(\pi_n(a))| \geq \]
\[ |\lambda_k| |\varphi_k((\pi_n(x))^k)| - \sum_{i=1}^{k-1} |\lambda_i||\varphi_k((\pi_n(x))^i)| \geq \]
\[ |\lambda_k| \cdot 1 - \sum_{i=1}^{k-1} |\lambda_i||\varphi_k||((\pi_n(x))^i)| > k|\varphi_k| \]
for all \( k \in \mathbb{N} \). Since each \( \varphi_k \) is a non-zero functional, this forces \( \| a \|_n \) to be infinite, a contradiction. Lemma 2.3 then shows that \( \dot{x} \) must be locally nilpotent.

Suppose that some power \( x^{m_0} \) is contained in \( \bigcap_{n=1}^{\infty} x^n A \). Since
\[ x^{m_0} \in \bigcap_{n=1}^{\infty} x^n A \subseteq x^{m_0+1} A \subseteq x^{m_0} A^i \]
this means that \( x \) has finite closed descent, which is the the first alternative of (vii.). Suppose that \( x \) is locally nilpotent and let \( n \in \mathbb{N} \). Then there exists \( m \in \mathbb{N} \) such that \( \| x^m \|_n = 0 \). This means that
\[ x^{m_0} \in \bigcap_{n=1}^{\infty} x^n A \subseteq I_n \]
Since \( n \) was arbitrary, we conclude that \( x^{m_0} \in I_n \) for all \( n \in \mathbb{N} \), and that \( x \) must be nilpotent, a contradiction to (i.). Hence \( x \) must be non-locally nilpotent if some power \( x^{m_0} \) is contained in \( \bigcap_{n=1}^{\infty} x^n A \).

But then if \( A^i \) were a commutative radical Fréchet algebra with identity adjoined, Corollary 2.5 would imply that \( x \) is \textit{not} weakly almost prime, contradicting assertion (vi.) proved above. Therefore, if the first assertion of (vii.) holds, it must be the case that \( A^i \) cannot be a commutative radical Fréchet algebra with identity adjoined.

If \( x \) does not have finite closed descent we have already shown that \( \dot{x} \) is locally nilpotent, and that \( \sigma(\dot{x}) = \{0\} \). If \( A \) were a Banach algebra then \( \dot{x} \) would have to be nilpotent, that is
\[ x^{m_0} \in \bigcap_{n=1}^{\infty} x^n A \]
for some \( m_0 \in \mathbb{N} \). However this forces \( x \) to have finite closed descent as we noted above. Consequently, in the second alternative of (vii.) it must be the case that \( A \) is \textit{not} a Banach algebra and \( \dot{x} \) is locally nilpotent but non-nilpotent. This finishes the proof of (vii.) and the proof of the Proposition. \[ \blacksquare \]
§3. Derivations on Fréchet Algebras. In this section we investigate further the relationship between derivations and local power series quotients. We do not assume that a derivation is continuous. We shall not at first assume that the algebra is commutative and consequently the remark following Definition 2.1 is operative.

The first part of this section will be reminiscent of the work of Johnson and Sinclair (see [Johnson] and [Johnson-Sinclair]) which showed that the discontinuity of a derivation on a Banach algebra is, in some sense, localized on finitely many primitive ideals. From this reduction, one can prove that if there exists a (necessarily discontinuous) derivation on a (necessarily non-commutative, see [Thomas2][Thomas3]) Banach algebra then there exists a (necessarily non-commutative) radical Banach algebra with a local power series quotient. If it could then be shown that no such radical Banach algebra exists, then every derivation on a Banach algebra would leave the primitive ideals invariant, an assertion also known as the non-commutative Singer-Wermer conjecture. It is currently unproven.

In the more general setting of Fréchet algebras, one can even find continuous derivations which do not leave the radical invariant, for example, formal differentiation on the Fréchet algebra of formal power series over the complex field $\mathcal{C}[X]$ (the Jacobson radical is simply $X \mathcal{C}[[X]]$). So it is not at first clear how the Banach algebra results should generalize to the situation of Fréchet algebras. Our final goal is Theorem 4.8 which shows that, except for the fact that Fréchet algebras can have locally nilpotent, non-nilpotent elements, the situation is very similar to that of Banach algebras.

We will start our reduction using a combination of representation theory together with the theory of automatic continuity. A reference for the general automatic continuity for linear functions on Fréchet spaces is [Thomas1]. In actuality, the automatic continuity results for Fréchet spaces correspond nicely to the results for Banach spaces (see [Sinclair]) except that closures have to be considered with respect to each seminorm. However, when one considers the strictly irreducible representations of a Fréchet algebra $\mathcal{A}$ many complications occur. Even when $\mathcal{A}$ is commutative, in which case the finite dimensional representations are 1-dimensional, corresponding to characters $\varphi$ from $\mathcal{A}$ into the complex field, it is not immediate that $\varphi$ must be continuous and the kernel of $\varphi$ (which is a maximal
ideal) therefore closed. This is the classical Michael’s problem (see [Michael] and, for a reduction of the problem, [Esterle]).

We will definitely have to make certain assumptions which we collect in Definition 3.1. We will observe the customary notational conventions. We let \( \text{irred}(\mathcal{A}) \) be the full set of non-isomorphic strictly irreducible representations of \( \mathcal{A} \) as linear functions on some complex vector space \( X \). We let \( \dim(\pi) \) be the dimension of \( X \) over the complex field, and for \( n \in \mathbb{N} \) we let \( \mathcal{M}_n \) be the algebra of \( n \) by \( n \) matrices with entries in the complex field \( \mathbb{C} \). So if \( \dim(\pi) = n \in \mathbb{N} \) then \( \pi \) is a homomorphism from \( \mathcal{A} \) into \( \mathcal{M}_n \). We say an ideal \( P \) is \emph{primitive} if it is the kernel of some strictly irreducible representation \( \pi \) and we write \( P = \ker(\pi) \). We let \( \text{prim}(\mathcal{A}) \) be the set of all primitive ideals of \( \mathcal{A} \) so that

\[
\text{prim}(\mathcal{A}) = \{ P \mid P = \ker(\pi) \text{ for some } \pi \in \text{irred}(\mathcal{A}) \}
\]

Finally, we define the (Jacobson) radical to be the intersection of all primitive ideals and denote it by \( \text{rad}(\mathcal{A}) \).

**Definition 3.1.** Let \( \mathcal{A} \) be a (possibly non-commutative) Fréchet algebra over the complex field. We define the subset \( \text{Irred}(\mathcal{A}) \subseteq \text{irred}(\mathcal{A}) \) to consist of those \( \pi \in \text{irred}(\mathcal{A}) \) such that

(3.1a) if \( X \) is the complex vector space on which \( \pi(\mathcal{A}) \) acts then \( X \) can be given a Banach space topology for which \( \pi(a) \) is a bounded linear operator for each \( a \in \mathcal{A} \) and for which the map \( a \to \pi(a) \) is continuous.

(3.1b) in the case that \( \mathcal{A} \) is not a Banach algebra we also require that if \( \pi \in \text{Irred}(\mathcal{A}) \) then \( n = \dim(\pi) < \infty \) so that \( \pi \) maps \( \mathcal{A} \) into \( \mathcal{M}_n \).

We define the subset \( \text{Prim}(\mathcal{A}) \subseteq \text{prim}(\mathcal{A}) \) to consist of those primitive ideals \( P \in \text{prim}(\mathcal{A}) \) such that

(3.1c) \( P \) is the kernel of some \( \pi \in \text{Irred}(\mathcal{A}) \) so, in particular, \( P \) must be closed.

We note for a Banach algebra \( \mathcal{A} \) (see [Rickart] or [Bonsall-Duncan]), \( \text{prim}(\mathcal{A}) = \text{Prim}(\mathcal{A}) \) and \( \text{irred}(\mathcal{A}) = \text{Irred}(\mathcal{A}) \). In addition, for every element \( a \in \mathcal{A} \) we have

\[
\sigma(a) \subseteq \left( \bigcup_{\pi \in \text{Irred}(\mathcal{A})} \sigma(\pi(a)) \right) \cup \{0\}
\]

The fact that the complex field is algebraically closed together with the Jacobson density theorem give us the following.
Lemma 3.2. Let \( \mathcal{A} \) be a (possibly non-commutative) Fréchet algebra (over the complex field). Then if \( \pi \in \text{Irred}(\mathcal{A}) \) and \( P = \ker(\pi) \) we have

(i.) \( \mathcal{A}/P \cong \mathcal{M}_{\dim(\pi)} \), and

(ii.) \( \mathcal{A}/P \) is a Banach algebra under one of its quotient seminorms.

Proof. We know that \( \mathcal{A}/P \) is isomorphic to a (finite dimensional semisimple) full matrix algebra with entries in a division ring \( \mathcal{D} \) whose centralizer contains the complex field \( \mathbb{C} \). However, any division ring \( \mathcal{D} \) which is finite dimensional over the complex field and contains the complex field in its center (this lets out the quaternions) would be algebraic over the complex field. Since the complex field is algebraically closed this forces \( \mathcal{D} = \mathbb{C} \). Hence the standard Wedderburn theory (basically an application of the Jacobson density theorem) shows that \( \pi \) maps \( \mathcal{A} \) onto \( \mathcal{M}_n \) where \( n = \dim(\pi) \), and \( \mathcal{A}/P \cong \mathcal{M}_n \), proving (i.).

For the second assertion note that \( \mathcal{A}/P \) must satisfy the descending chain condition. Let a “dot” denote the coset in \( \mathcal{A}/P \). Then

\[
\{ \dot{a} \in \mathcal{A}/P \mid \| a + P \|_n = 0 \}
\]

is certainly a descending chain (indexed on \( n \)) of closed ideals whose intersection is the zero coset. Consequently, one of the seminorms must actually be a norm, proving (ii.)

\[ \blacksquare \]

As a consequence of Lemma 3.2(ii.) we will simply use \( \| \cdot \| \) to denote the Banach algebra norm on \( \mathcal{A}/P \).

Definition 3.3. A derivation on a (possibly non-commutative) Fréchet algebra \( \mathcal{A} \) is a linear map \( D \) from \( \mathcal{A} \) to itself satisfying

\[
D(ab) = a(Db) + (Da)b
\]

for all \( a, b \) in \( \mathcal{A} \). We emphasize (again) that we do not require \( D \) to be continuous. Note that a derivation \( D \) extends to \( \mathcal{A}^2 \) in an obvious way by defining (if necessary) \( D1 = 0 \). We now state a well known result from the theory of automatic continuity.

Lemma 3.4. Let \( D \) be a (possibly discontinuous) derivation on a Fréchet algebra \( \mathcal{A} \). Let \( I \) be a closed ideal of \( \mathcal{A} \) and let \( Q_I \) denote the continuous canonical quotient map from \( \mathcal{A} \) onto \( \mathcal{A}/I \). For each \( k \in \mathbb{N} \) define the separating subspace of
$D^k$ as follows

$$S(D^k) \equiv \{ z \in \mathcal{A} \mid \exists x_i \to 0 \text{ in } \mathcal{A} \text{ and } D^k(x_i) \to z \}$$

If the codimension $[\mathcal{A} : I]$ is finite then the following are equivalent:

(i.) $Q_I D^k$ is continuous on $I$ for all $k \in \mathbb{N}$.

(ii.) $Q_I D^k$ is continuous on $\mathcal{A}$ for all $k \in \mathbb{N}$.

(iii.) $S(D^k) \subseteq I$ for all $k \in \mathbb{N}$.

In the special case where $\mathcal{A}$ is actually a Banach algebra, any one of the above implies the following condition:

(iv.) If $I \in \text{Prim}(\mathcal{A})$ then $D(I) \subseteq I$.

**Proof.** Assume that condition (i.) holds. Since a finite extension of a continuous linear mapping on a closed subspace is continuous it is clear the condition (ii.) must hold also. Clearly condition (ii.) implies condition (i.). Hence they are equivalent.

Assume that condition (ii.) holds. It is fundamental (see [Thomas1, discussion on results (1)-(3), pages 518-519]) that $Q_I D^k$ will be continuous on $\mathcal{A}$ precisely when $Q_I(S(D^k)) = \{0\}$. This will happen precisely when $S(D^k) \subseteq I$. Hence, condition (ii.) is equivalent to condition (iii.).

Suppose $\mathcal{A}$ is a Banach algebra and assume that condition (ii.) above holds. An application of [Thomas3, Lemma 1.1] shows that there exists a constant $C > 0$ such that

$$\|Q_I D^k\| \leq C^k$$

for all $k \in \mathbb{N}$ (it is not necessary for $I$ to be primitive for this). Now, given that $I$ is primitive, apply [Thomas3, Lemma 1.2] and this shows that $D(I) \subseteq I$. ■

We have a partial converse to the above rate of growth condition on $Q_I D^k$ as follows.

**Lemma 3.5.** Let $D$ be a (possibly discontinuous) derivation on a Fréchet algebra $\mathcal{A}$. Let $P \in \text{Prim}(\mathcal{A})$. Since our assumptions imply that $P$ is closed with finite codimension let $Q_P$ denote the continuous canonical quotient map from $\mathcal{A}$ onto $\mathcal{A}/P$. If, for each $a \in P$ there exists $C_a > 0$ such that $\|Q_P D^k a\|^{1/k} \leq C_a$ for
all $k \in \mathbb{N}$ then there exist globals $C > 0$ and $M \in \mathbb{N}$ such that
\[
\|Q_P D^k a\| \leq C^k \|a\|_M
\]
for all $a \in P$. In addition, $Q_P D^k$ is continuous on $\mathcal{A}$ for all $k \in \mathbb{N}$ and $D(P) \subseteq P$.

**Proof.** For each $N \in \mathbb{N}$ define
\[
X_N \equiv \{a \in P \mid \|Q_P D^k a\|^{1/k} \leq N \text{ for all } k \in \mathbb{N}\}
\]
It is easily checked that $X_N$ is a closed subset of $P$ for each $N \in \mathbb{N}$ and that
\[
\bigcup_{N=1}^{\infty} X_N = P.
\]
Since a complete metric space is not of the first category there must exist some $N_0$ which contains an open neighborhood. Therefore, it follows that
\[
X_N - X_N = \{a_1 - a_2 \mid a_1, a_2 \in X_N\}
\]
contains an open neighborhood at 0. Hence, there exists a seminorm $\|\cdot\|_{M_0}$ such that $\|a\|_{M_0} < 1$ implies that $a = a_1 + a_2$ for some $a_1, a_2 \in X_N$. Linearity of $D$ and subadditivity of the norm then show that
\[
\|Q_P D^k a\| = \|Q_P D^k a_1 + Q_P D^k a_2\|
\leq N_0^k + N_0^k = 2N_0^k \leq (2N_0)^k
\]
We conclude that if $a \in P$ with $\|a\|_{M_0} < 1$ then $\|Q_P D^k a\|^{1/k} \leq (2N_0)$. Now suppose that $a \in P$ but $\|a\|_{M_0} \geq 1$. Let $\delta > 0$ and note that if $b = a/(\|a\|_{M_0} + \delta)$ then $\|Q_P D^k b\|^{1/k}_{M_0} \leq (2N_0)$ which implies that $\|Q_P D^k a\|_{M_0} \leq (2N_0)^k (\|a\|_{M_0} + \delta)$. Letting $M = M_0$ , $C = (2N_0)$ , and letting $\delta \to 0$ establishes the first part of assertion (i.) as well as the continuity of $Q_P D^k$ on $P$. Lemma 3.4 shows that $Q_P D^k$ is continuous on all of $\mathcal{A}$.

It remains to show that $D(P) \subseteq P$. The argument is virtually identical to the proof of Lemma 1.2 of [Thomas3]. We note that $\mathcal{A}/P$ is a finite dimensional primitive Banach algebra with one faithful irreducible representation. It is certainly semi-simple. Furthermore, the Leibniz rule shows that $(D(P) + P)/P$ is an ideal of $\mathcal{A}/P$. It is either trivial or the whole algebra. Let $a \in P$ and note that
\[
D^k(a^k) \in k!(Da)^k + P
\]
Therefore we can compute
\[
(k!)^{1/k} \|Q_P D a\|^{1/k} = \|Q_P D^k(a^k)\|^{1/k} \leq C\|a^k\|_M^{1/k}
\]
Since \( \lim_{k \to \infty} \| a^k \|^{1/k} = \rho_M(a) < \infty \) and since \( \lim_{k \to \infty} (k!)^{1/k} = \infty \), it must be the case that \( \lim_{k \to \infty} \| (Q_PD\alpha)^k \|^{1/k} = 0 \) for all \( \alpha \in P \). This means that \( (D(P) + P)/P \) is a quasinilpotent (actually nilpotent due to the finite dimensionality) ideal in the semi-simple Banach algebra \( \mathcal{A}/P \). Hence it must be the trivial ideal and thus \( D(P) \subseteq P \). 

Before going further we wish to introduce a third example.

**Example 3.6.** Let \( \mathcal{A} = \mathcal{O}(\Delta) \) where

\[
\mathcal{O}(\Delta) \equiv \{ f \text{ analytic on the open unit disk } \Delta \}
\]

with pointwise multiplication. This commutative Fréchet algebra is unital and we can define seminorms

\[
\| f \|_n = \sup\{|f(\zeta)| \, | \zeta| \leq (1 - 1/n)\}
\]

for each \( n \in \mathbb{N} \). The resulting topology is, of course, the topology of uniform convergence on compact subsets of the open unit disk \( \Delta \). Consequently, \( \mathcal{B}_n \) is essentially the Banach algebra of functions analytic on \( \{ \zeta \mid |\zeta| < (1 - 1/n) \} \) and continuous on \( \{ \zeta \mid |\zeta| \leq (1 - 1/n) \} \). The projection \( \pi_n \) is dense but not onto \( \mathcal{B}_n \).

We have a natural continuous derivation \( D \) on \( \mathcal{O}(\Delta) \), namely \( Df = f' \). Let \( \zeta_0 \in \Delta \) be fixed. Then \( \pi f = f(\zeta_0) \) is a strictly irreducible (one-dimensional) representation of \( \mathcal{O}(\Delta) \) and \( P = \text{kernel}(\pi) = \{ f \in \mathcal{O}(\Delta) \mid f(\zeta_0) = 0 \} \). Clearly \( D(P) \not\subseteq P \) for this derivation and primitive ideal. However, we can compute

\[
\| Q_P D^k f \| = |Q_P D^k f| = |f^{(k)}(\zeta_0)|
\]

\[
= \frac{k!}{2\pi} \left| \int_{\Gamma} \frac{f(\zeta)}{(\zeta - \zeta_0)^{k+1}} d\zeta \right|
\]

(where the “\( \pi \)” above is the constant 3.14159... ) for \( f \in P, k \in \mathbb{N} \), and \( \Gamma \) a contour about \( \zeta_0 \). If we choose \( M \in \mathbb{N} \) sufficiently large so that \( |\zeta_0| + 1/M < (1 - 1/M) \), and let \( \Gamma \) be a circle oriented counterclockwise about \( \zeta_0 \) of radius \( 1/M \), then the above will be bounded by \( (k!/(2\pi)) \| f \|_{\mathcal{M}} (1/M)^{k+1} \). We have therefore shown that there is a constant \( C > 0 \) so that

\[
|Q_P D^k f| \leq C^k (k!) \| f \|_{\mathcal{M}}
\]
for all $f \in P$. This does not however, give a formal power series quotient for $\mathcal{O}(\Delta)$ based at any $f_0$ for the following reason. If $f_0 \in \mathcal{O}(\Delta)$ with $f_0 \mathcal{O}(\Delta)$ a proper ideal then $f_0$ must vanish at at least one point $\zeta_0$ of $\Delta$. Consequently, the set of elements of infinite height $\cap_{m=1}^{\infty} f_0^m \mathcal{O}(\Delta)$ must be $\{0\}$ since no analytic function in $\mathcal{O}(\Delta)$ can have a zero of infinite order at $\zeta_0 \in \Delta$. Also, each of the range space closures satisfy

$$f_0^k \mathcal{O}(\Delta) \neq f_0^{k+1} \mathcal{O}(\Delta),$$

for all $k \in \mathbb{N}$, so that $f$ cannot have finite closed descent. If $\mathcal{O}(\Delta)$ had a formal power series quotient based at $f_0$ the kernel would be trivial and then Proposition 2.6(vii.) would imply that the spectrum of $f_0$ in $\mathcal{O}(\Delta)$ is simply $\{0\}$ which is nonsense since

$$\sigma(f_0) = \{f_0(\zeta) \mid \zeta \in \Delta\}$$

and $f_0 \neq 0$. There is, of course, a homomorphism from $\mathcal{O}(\Delta)$ into $\mathfrak{g}[[X]]$ via the power series expansion about $\zeta_0$:

$$f \rightarrow \sum_{k=0}^{\infty} \left( \frac{f^{(k)}(\zeta_0)}{k!} \right) X^k$$

$$= \sum_{k=0}^{\infty} \left( \frac{\pi(D^k f)}{k!} \right) X^k$$

The essential point is that such a homomorphism cannot be onto.

Recalling our previous example (1.3) $\mathcal{A} = C^{\infty}(\mathbb{R})$ (where we have many local power series quotients), we see that the condition we had in $\mathcal{O}(\Delta)$, namely

$$|Q_rD^k f| \leq C^k(k!)\|f\|_M$$

certainly fails in $C^{\infty}(\mathbb{R})$ because there are a plentiful supply of functions whose Taylor series expansions do not have a positive radius of convergence.

The classical uses of derivatives on function spaces depend upon derivatives not leaving primitive ideals invariant. With this and the preceding examples in mind we make the following definition which identifies four possible cases of behavior.

**Definition 3.7.** Let $D$ be a (possibly discontinuous) derivation on a Fréchet algebra $\mathcal{A}$. Let $P \in \text{Prim}(\mathcal{A})$.

i. If $D(P) \subseteq P$ we say that $D$ is invariant at $P$. 

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ii. If there exist $C > 0$ and $M \in \mathbb{N}$ such that

$$
\|Q_P D^k a\| \leq C^k(k!)\|a\|_M
$$

for every $a \in P$ we say that $D$ is analytic at $P$.

iii. If $D$ is not analytic at $P$ but $Q_P D^k$ is continuous for every $k \in \mathbb{N}$ we say that $D$ is continuously differentiable at $P$.

iv. If $Q_P D^k$ is discontinuous for some $k \in \mathbb{N}$ we say that $D$ is singular at $P$.

Note that case (i.) can, in general, overlap with the other cases, although cases (ii.), (iii.), and (iv.) are distinct. The unproven non-commutative Singer-Wermer conjecture states that if $\mathcal{A}$ is a Banach algebra then case (i.) is always true.

The following lemma is well known in the case of a Banach algebra (see [Thomas3, Lemma 1.1 and 1.2]). It is also well known in the case of a Jordan-Banach algebra (see [Villena, Theorem 6]). We require a generalization to Fréchet algebras.

**Lemma 3.8.** Let $D$ be a (possibly discontinuous) derivation on a Fréchet algebra $\mathcal{A}$. Let $P \in \text{Prim}(\mathcal{A})$. Define

$$
J_P \equiv \{x \in P \mid D^k x \in P \text{ for all } k \in \mathbb{N}\}
$$

Then $J_P$ is a $D$-invariant ideal contained in $P$.

Suppose further that $D$ is continuously differentiable at $P$, i.e., $Q_P D^k$ is continuous on $P$ for all $k \in \mathbb{N}$. Then $J_P$ is a closed ideal and $\mathcal{S}(D^k) \subseteq J_P$ for all $k \in \mathbb{N}$.

Finally, if $D$ is continuously differentiable at $P$ and $\mathcal{A}$ is actually a Banach algebra, then the following additional conditions hold:

(i.) There exists $C > 0$ such that $\|Q_P D^k\| \leq C^k$ for all $k \in \mathbb{N}$, and

(ii.) $D(P) \subseteq P$.

**Proof.** The Leibniz rule implies that

$$
D^n(yx) = (D^n y)x + \sum_{i=0}^{n-1} \binom{n}{i} (D^i y) D^{n-i} x
$$

for $y \in \mathcal{A}$, $x \in P$, and $k \in \mathbb{N}$. Hence $J_P$ is a left ideal. An analogous argument shows that it is a right ideal. It is closed since $Q_P D^k$ is continuous for each
$k \in \mathbb{N}$. In addition, the fact that $Q_P D^k$ is continuous on $P$ for all $k \in \mathbb{N}$ shows that $Q_P D^k(S(D^k)) = \{0\}$ or, equivalently, that $S(D^k) \subseteq P$ for all $k \in \mathbb{N}$. We claim that actually $S(D^k) \subseteq J_P$ for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and let $x_i \to 0$ in $P$ with $D^k x_i \to z \in P$. Apply the continuous operator $Q_P D^n$ in order to obtain $Q_P D^{n+k} x_i \to Q_P D^n z$. But since $Q_P D^{n+k}$ is also continuous on $P$, we must have that $Q_P D^{n+k} x_i \to 0$. Hence we see that $z \in P$ and $Q_P D^n z = 0$, or equivalently, $D^n z \in P$ for all $n \in \mathbb{N}$. This shows that $z \in J_P$. Since $z$ was an arbitrary element of $S(D^k)$ this shows that $S(D^k) \subseteq J_P$. Since the argument applies for every $k \in \mathbb{N}$ the lemma has been proved in the case of a Fréchet space.

If $\mathcal{A}$ is a Banach algebra then [Thomas3, Lemma 1.1] implies condition (i.) and [Thomas3, Lemma 1.2] implies condition (ii.). □

We now investigate the how often the singular case can occur. If $Q_P D^k$ is discontinuous for some $k \in \mathbb{N}$ then there is a first such $k = k_0 \in \mathbb{N}$ and $Q_P D^\ell$ is continuous from $\mathcal{A}$ to $\mathcal{A}/P$ for $\ell = 0, 1, \ldots, (k_0 - 1)$.

Furthermore, since $\mathcal{A}/P$ is finite dimensional and isomorphic to an algebra of matrices there is a smallest $m_0 \in \mathbb{N}$ such that

$$\{ (\mathcal{A}/P, \| \cdot + P \|_m) \mid m \geq m_0 \}$$

is a set of continuously isomorphic Banach algebras. Hence there is some $q_0 \in \mathbb{N}$ and $C_m > 0$ for $m = m_0, m_0 + 1, \ldots$ such that for all $x \in \mathcal{A}$ we have

$$\| Q_P D^\ell (x) + P \|_m \leq C_m \| x \|_{q_0}$$

for $m \geq m_0$ and $\ell = 0, 1, \ldots, (k_0 - 1)$. We formalize this as a definition.

**Definition 3.9.** Let $D$ be a derivation on a Fréchet algebra $\mathcal{A}$. Let $P \in \text{Prim} (\mathcal{A})$ and let $D$ be singular at $P$. Let $k$ be the first power such that $Q_P D^k$ is discontinuous. Let $q$ be the smallest index for which $Q_P D^\ell$ for $\ell = 0, 1, \ldots, (k - 1)$ is continuous as a linear map from $(\mathcal{A}, \| \cdot \|_q)$ into $\mathcal{A}/P$. We call $q$ the index of $P$ and we write $q = \text{index}(P)$.

In addition, for each $q \in \mathbb{N}$ define the $q$th-exceptional set $\mathcal{E}_q(D)$ as follows

$$\mathcal{E}_q(D) \equiv \{ P \in \text{Prim} (\mathcal{A}) \mid D \text{ is singular at } P \text{ and } \text{index}(P) \leq q \}$$

This definition does not depend on the choice of norm on $\mathcal{A}/P$ since this Banach algebra is finite dimensional.
Our goal is to show that each $\mathcal{E}_q(D)$ is a finite (or empty) set, thereby reducing the problem to dealing with countably many points at which $D$ is singular. This result is well known in the case of a Banach algebra (see [Thomas3, Proposition 1.10] which is strongly indebted to [Johnson-Sinclair]). It is also well known in the case of a Jordan-Banach algebra (see [Villena, Theorem 7]). The generalization to Fréchet algebras requires using the somewhat weaker conclusions of the automatic continuity for linear functions on Fréchet spaces (see [Thomas1]), but this is sufficient. However, something has been lost (in the non-commutative case) in that we have had to exclude consideration of continuous infinite dimensional strictly irreducible representations of (non-Banach) Fréchet algebras from Irred($\mathcal{A}$) in Definition 3.1.

The most general result known for handling infinite dimensional strictly irreducible representations is due to Villena [Villena, Corollary 1] and depends on the concept of a sliding hump sequence pair:

**Definition 3.10.** (A. R. Villena) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of linear operators on a vector space $X$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $X$ itself. The couple ($\{a_n\}, \{x_n\}$) is said to be a sliding hump sequence pair if the following properties are satisfied:

\[ a_n \ldots a_2 a_1 x_n \neq 0 \]
\[ a_{n+1} a_n \ldots a_2 a_1 x_n = 0 \]
\[ a_1 a_2 \ldots a_n \neq 0 \]

for all $n \in \mathbb{N}$.

**Corollary 3.11.** (A. R. Villena) Let $\mathcal{A}$ be an associative algebra of continuous linear operators acting irreducibly on an infinite dimensional complex normed space $X$. Assume that $J$ is a linear subspace of the bounded linear operators, $BL(X)$, on $X$ which contains $\mathcal{A}$ and is endowed with a complete norm which makes the inclusion $J \rightarrow BL(X)$ continuous. Then there exists a sliding hump sequence pair in $J$.

In our situation, even if we assume that all primitive ideals are closed it is not clear how to handle the case of a primitive ideal $P$ whose codimension $[\mathcal{A} : P]$ is infinite since it is unlikely that the vector space on which the representation acts is normable.
We therefore restrict our attention to \( \text{Irred}(\mathcal{A}) \) and proceed with the following two lemmas which are essentially due to Johnson and Sinclair [Johnson-Sinclair, Lemma 3.1 and Lemma 3.2] with modifications to handle the Fréchet algebra situation.

**Lemma 3.12.** (Johnson and Sinclair) Let \( \{\pi_i\}_{i=1}^j \subseteq \text{Irred}(\mathcal{A}) \) be non-equivalent strictly irreducible representations of a Fréchet algebra \( \mathcal{A} \) over the complex field on (respectively) finite dimensional complex vector spaces \( \{X_i\}_{i=1}^j \). Let \( \dim(X_i) = n_i \), let \( I_i \) be the identity matrix in \( \mathcal{M}_{n_i} \), and let \( P_i = \text{kernel}(\pi_i) \) for \( i = 1, 2, \ldots, j \). Then

\[
\mathcal{A}/(P_1 \cap P_2 \cap \ldots \cap P_j) \cong (\mathcal{A}/P_1) \oplus (\mathcal{A}/P_2) \oplus \ldots \oplus (\mathcal{A}/P_j)
\cong \mathcal{M}_{n_1} \oplus \mathcal{M}_{n_2} \oplus \ldots \oplus \mathcal{M}_{n_j},
\]

and there exists \( a \in \mathcal{A} \) such that \( \pi_i(a) = 0 \) for \( i < j \) but \( \pi_j(a) = I_j \). In addition, if each \( \pi_i \) is discontinuous as a linear map from \( (\mathcal{A}, \| \cdot \|_q) \) into \( \mathcal{M}_{n_i} \) for \( i = 1, 2, \ldots, j \) then given \( \epsilon > 0 \) there exists \( a \in \mathcal{A} \) such that \( \pi_i(a) = 0 \) for \( i < j \), \( \pi_j(a) = I_j \), and \( \|a\|_q < \epsilon \).

**Proof.** Although [Johnson-Sinclair, Lemma 3.1] is stated for Banach algebras, the proof goes through verbatim for Fréchet algebras due to the cofiniteness of the primitive ideals. Therefore the assertion of the isomorphism above holds and the isomorphism can be implemented via the map

\[
\varphi : a \mapsto (\pi_1(a), \pi_2(a), \ldots, \pi_j(a))
\]

Hence there exists an element \( a_0 \in \mathcal{A} \) with \( \pi_i(a_0) = 0 \) for \( i < j \) but \( \pi_j(a_0) = I_j \). It is clear that \( \text{kernel}(\varphi) = P_1 \cap P_2 \cap \ldots \cap P_j \) and that \( \text{kernel}(\varphi) \) is closed since the primitive ideals \( P_i \) are closed for \( i = 1, 2, \ldots, j \) (here we are using the fact that each \( \pi_i \in \text{Irred}(\mathcal{A}) \)). Suppose that the closure in the \( q \)-th seminorm, \( \overline{\text{kernel}(\varphi)}^q \) is not all of \( \mathcal{A} \). The image of \( \overline{\text{kernel}(\varphi)}^q \) must then be a proper ideal of \( \mathcal{M}_{n_1} \oplus \mathcal{M}_{n_2} \oplus \ldots \oplus \mathcal{M}_{n_j} \), which is semi-simple. It must then be identically zero in some coordinate, say \( \mathcal{M}_{n_i} \), since the proper closed ideals of \( \mathcal{M}_{n_1} \oplus \mathcal{M}_{n_2} \oplus \ldots \oplus \mathcal{M}_{n_j} \) are precisely products selected from a proper subset of \( \{\mathcal{M}_{n_1}, \mathcal{M}_{n_2}, \ldots, \mathcal{M}_{n_j}\} \) together with \{0\} in the remaining factors. This means that \( \pi_i(\overline{\text{kernel}(\varphi)}^q) = \{0\} \), and, hence, \( \pi_i \) factors through the finite dimensional normed (and hence, Banach) algebra \( \mathcal{A}/\overline{\text{kernel}(\varphi)}^q, \| \cdot \|_q \) as some continuous map \( \hat{\varphi} \). This
contradicts the fact that \( \pi_i \) is discontinuous from \((\mathcal{A}, \| \cdot \|_q)\). We have therefore shown that

\[
\overline{\text{kernel}(\varphi)} = \mathcal{A}
\]

The kernel of \( \varphi \) is therefore \( \| \cdot \|_q \)-dense in \( \mathcal{A} \) and we can pick an element \( a \) in the coset \((a_0 + \text{kernel}(\varphi))\) whose \( q \)-seminorm value is as small as desired and this element will satisfy the requirements of the lemma.

\[\square\]

**Lemma 3.13.** (Johnson and Sinclair) Let \( \{\pi_i\}_{i=1}^{\infty} \in \text{Irred}(\mathcal{A}) \) be non-equivalent strictly irreducible representations of a Fréchet algebra \( \mathcal{A} \) over the complex field on (respectively) finite dimensional complex vector spaces \( X_i \), for all \( i \in \mathbb{N} \). Then there exists a sequence \( \{a_i\}_{i=1}^{\infty} \subseteq \mathcal{A} \) such that

\[
\pi_n(a_m) = 0
\]

whenever \( m > n \), but

\[
\pi_n(a_m) \text{ is regular in } \mathcal{M}_{\dim(X_n)} \text{ for } n \geq m
\]

**Proof.** Let \( d \) be any metric giving seminorm-wise convergence in \( \mathcal{A} \) (for example, \( d(x,y) = \sum_{k=1}^{\infty} 2^{-k}(\|x-y\|_k)/(1+\|x-y\|_k) \)). Let \( \dim(X_i) = n_i \) for \( i \in \mathbb{N} \). Let \( I_i \) denote the identity matrix in \( \mathcal{M}_{n_i} \). Temporarily fix \( k \in \mathbb{N} \). Choose \( z_j \) for \( j = k, k+1, \ldots \) such that \( \pi_i(z_j) = 0 \) for \( 1 \leq i < j \) and \( \pi_j(z_j) = I_j \). Lemma 3.12 guarantees that such \( z_j \) exist for \( j = k, k+1, \ldots \). Next choose a sequence of of positive reals \( \{\epsilon_j\}_{j=k}^{\infty} \) by induction so that the two conditions

\[
d(0, \epsilon_j z_j) < 2^{-j} \quad \text{and} \quad \pi_j(\epsilon_k z_k + \ldots + \epsilon_j z_j) = \pi_j(\epsilon_k z_k + \ldots + \epsilon_{j-1} z_{j-1}) + \epsilon_j I_j \text{ is regular}
\]

are satisfied. This is possible because

\[
\pi_j(\epsilon_k z_k + \ldots + \epsilon_{j-1} z_{j-1}) + \lambda I_j
\]

is singular in \( \mathcal{M}_{n_j} \) for only a finite number of values of \( \lambda \). It is routine to check that \( y_k = \sum_{j=k}^{\infty} \epsilon_j z_j \) has the necessary properties.

\[\square\]

**Proposition 3.14.** Let \( D \) be a (possibly discontinuous) derivation on a Fréchet algebra \( \mathcal{A} \). Then for each \( q \in \mathbb{N} \) the exceptional set \( \mathcal{E}_q(D) \) is either empty or a finite set.
Proof. Suppose that the result fails for some $q \in \mathbb{N}$. For each primitive ideal $P \in \text{Prim}(\mathcal{A})$ fix a quotient norm on $\mathcal{A}/P$ (since all such norms are equivalent) and denote it by $\| \cdot + P \|$ (with no subscript). We can then find a sequence of non-equivalent strictly irreducible representations $\{\pi_i\}_{i=1}^{\infty} \subseteq \text{Irred}(\mathcal{A})$ with primitive ideals $\{P_i = \ker(\pi_i)\}_{i=1}^{\infty} \subseteq \text{Prim}(\mathcal{A})$ such that there exists $k_i \in \mathbb{N}$ and $C_i > 0$ such that

$$\|D^{k_i}(x) + P_i\| \leq C_i\|x\|_q,$$

for all $x \in \mathcal{A}$ and $\ell < k_i$, but

$$Q_{P_i} D^{k_i} \text{ is discontinuous}$$

for $i = 1, 2, \ldots$. Apply Lemma 3.13 in order to obtain the sequence $\{a_i\}_{i=1}^{\infty}$. Let $x \in \mathcal{A}$ and compute

$$Q_{P_n} D^{k_n}(xa_m \ldots a_2 a_1) = (Q_{P_n} D^{k_n}(x))(Q_{P_n}(a_m \ldots a_2 a_1))$$

$$+ \sum_{\ell=0}^{k_n-1} \binom{k_n}{\ell} \left( (Q_{P_n} D^{\ell}(x))(Q_{P_n} D^{k_n-\ell}(a_m \ldots a_2 a_1)) \right)$$

for $m, n \in \mathbb{N}$. Note that the map $x \rightarrow Q_{P_n} D^{\ell}(x)$ is continuous for $\ell = 0, 1, \ldots, (k_n - 1)$ from $(\mathcal{A}, \| \cdot \|_q)$ into $\mathcal{A}/P_n$. If $m > n$ we have that $\pi_n(a_m) = 0$ so $Q_{P_n}(a_m \ldots a_2 a_1) = 0$ and the map $x \rightarrow Q_{P_n} D^{k_n}(xa_m \ldots a_2 a_1)$ is continuous from $(\mathcal{A}, \| \cdot \|_q)$ into $\mathcal{A}/P_n$. In particular, we can find a constant $C_n > 0$ such that both

$$\|Q_{P_n} D^{k_n-1}y\| \leq C_n\|y\|_q$$

and

$$\|Q_{P_n} D^{k_n}(ya_{n+1}a_n \ldots a_2 a_1)\| \leq C_n\|y\|_q$$

hold for all $y \in \mathcal{A}$. However, if $m = n$ since $\pi_n(a_n \ldots a_2 a_1)$ is regular in $\mathcal{M}_{\dim(\pi_n)}$ and since the map $x \rightarrow Q_{P_n} D^{k_n}(x)$ is discontinuous, it must be the case that the map $x \rightarrow Q_{P_n} D^{k_n}(xa_n \ldots a_2 a_1)$ is discontinuous.

The argument from now on is rather a standard one in the theory of automatic continuity. For each $n \in \mathbb{N}$ use the discontinuity of the map $x \rightarrow Q_{P_n} D^{k_n}(xa_n \ldots a_2 a_1)$ in order to choose $x_n \in \mathcal{A}$ so that $\|x_n\|_q < 2^{-n}$, and $\|x_na_n \ldots a_{r+1}a_r\|_q < 2^{-n}$ for all $r = 1, 2, \ldots, n$, but

$$\|D^{k_n}(x_na_n \ldots a_2 a_1) + P_n\| > nC_n + \|D^{k_n}(\sum_{i=1}^{n-1} x_ia_i \ldots a_2 a_1) + P_n\|$$

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where the constant $C_n$ is the one above. Let $x = \sum_{i=1}^{\infty} x_i a_i \ldots a_2 a_1 \in \mathcal{A}$ which is easily seen to converge. For each $n \in \mathbb{N}$ we can obtain a lower bound for $\|Dx\|_q$ as follows:

$$\|Dx\|_q \geq C_n^{-1} \|(Q_{P_n} D^{k_n-1})(Dx)\|$$

$$= C_n^{-1} \|D^{k_n} x + P_n\|$$

$$= C_n^{-1} \|D^{k_n} \left( \sum_{i=1}^{\infty} x_i a_i \ldots a_2 a_1\right) + P_n\|$$

$$\geq C_n^{-1} \left( \|D^{k_n} (x_n a_n \ldots a_2 a_1) + P_n\| - \|D^{k_n} \left( \sum_{i=n+1}^{\infty} x_i a_i \ldots a_2 a_1\right) + P_n\|\right)$$

$$\geq C_n^{-1} \left( nC_n - \|D^{k_n} \left( \sum_{i=n+1}^{\infty} x_i a_i \ldots a_{n+2}\right) (a_{n+1} \ldots a_2 a_1) + P_n\|\right)$$

Since $\|(\sum_{i=n+1}^{\infty} x_i a_i \ldots a_{n+2})\|_q \leq 1$ we have that

$$\|Dx\|_q \geq C_n^{-1} \left( nC_n - C_n\right) = (n - 1)$$

for all $n \in \mathbb{N}$, a contradiction. This establishes the proof of the proposition. \[\square\]

In order to go further we need to specialize to commutative Fréchet algebras $\mathcal{A}$ since we need some topology on $\text{Irred}(\mathcal{A})$ and, in the commutative case, we at least have the topology of pointwise convergence (which is the weak* topology in the case of Banach algebras) as functions on $\mathcal{A}$. We will continue to call this the weak* topology, even though it is not in general a locally compact topology on $\text{Irred}(\mathcal{A})$ when $\mathcal{A}$ is a Fréchet algebra. Since $\mathcal{A}$ is commutative, it is clear that all elements $\pi \in \text{Irred}(\mathcal{A})$ are multiplicative homomorphisms into the complex field.

**Lemma 3.15.** Suppose that $T$ and $\{T_i\}_{i=1}^{\infty}$ are continuous linear operators from a Fréchet space $(X, \|\cdot\|_n)$ to a Banach space $(Y, \|\cdot\|)$ satisfying

$$\lim_{i \to \infty} T_i x = T x$$

for all $x \in X$. Then there exist $p \in \mathbb{N}$ and $C \in \mathbb{R}^+$ such that

$$\|T_i x\| \leq C\|x\|_p$$

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for all $x \in X$ and $i \in \mathbb{N}$.

**Proof.** Since $T$ is continuous there exist $q \in \mathbb{N}$ and $K \in \mathbb{R}^+$ such that $\|Tx\| \leq K\|x\|_q$ for all $x \in X$. Since $\lim_{i \to \infty} T_i x = Tx$ for every $x \in X$ there exists $K_x$ such that $\|T_i x\| \leq K_x\|x\|_q$ for every $i \in \mathbb{N}$. Let $X_n = \{x \in X \mid K_x \leq n\}$. Then $\bigcup_{n=1}^{\infty} X_n = X$ and since $\|T_i x\| \leq n\|x\|_q$ is a closed condition, each $X_n$ is closed. Since $(X, \| \cdot \|_n)$ is a complete metric space there is some $X_N$ which contains an $(\epsilon, p)$-ball

$$B(x_0, \epsilon, p) = \{x \in X \mid \|x - x_0\|_p < \epsilon\} \subseteq X_N$$

and, by passing to a somewhat larger $N$, we may assume that $x_0 \in X_N$. Let $x$ be any element of $X$ with $\|x\|_p = \epsilon/2$. Then $\|T_i(x_0 + x) - x_0\|_p < \epsilon$ so

$$\|T_i x\| = \|T_i(x_0 + x) - T_i x_0\| \leq N\|x_0 + x\|_p + N\|x_0\|_p \leq 2N\left(1 + \frac{\|x_0\|_p}{\epsilon}\right)\epsilon$$

Hence, for any $x \in X$ we have $\|T_i x\| \leq C\|x\|_p$ where

$$C = 2N\left(1 + \frac{\|x_0\|_p}{\epsilon}\right)$$

This finishes the proof of the lemma. $\blacksquare$

**Lemma 3.16.** Let $D$ be a (possibly discontinuous) derivation on a commutative Fréchet algebra $\mathcal{A}$. Then the union of the exceptional sets $\bigcup_{q=1}^{\infty} \mathcal{E}_q(D)$ is sequentially isolated in the weak* topology.

**Proof.** Suppose that the result fails and there exists some pairwise disjoint sequence $\{\pi_i\}_i^{\infty} \subseteq \bigcup_{q=1}^{\infty} \mathcal{E}_q(D)$ and $\pi \in \bigcup_{q=1}^{\infty} \mathcal{E}_q(D)$ such that

$$\lim_{i \to \infty} \pi_i y = \pi y$$

for each $y \in \mathcal{A}$. Let $P = \text{kernel}(\pi)$. There exist $k, q \in \mathbb{N}$ such that $Q_P D^k$ is discontinuous, but

$$Q_P D^\ell : (\mathcal{A}, \| \cdot \|_q) \to \mathcal{A}/P \cong \mathbb{C}$$

is continuous for $\ell = 0, 1, 2, \ldots k - 1$. For each $i \in \mathbb{N}$ let $P_i = \text{kernel}(\pi_i)$. For each $i \in \mathbb{N}$ there exist $k_i, q_i \in \mathbb{N}$ such that $Q_{P_i} D^{k_i}$ is discontinuous, but

$$Q_{P_i} D^\ell : (\mathcal{A}, \| \cdot \|_{q_i}) \to \mathcal{A}/P_i \cong \mathbb{C}$$

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is continuous for $\ell = 0, 1, 2, \ldots k_i - 1$. We first claim that eventually $k_i \leq k$. If not, drop to a subsequence such that $k_i > k$ for all $i \in \mathbb{N}$. Then for all $i \in \mathbb{N}$ it would be the case that $Q_{P_i} S(D^k) = \{0\}$, and, hence, that $\pi_i(S(D^k)) = \{0\}$. Since

$$\lim_{i \to \infty} \pi_i z = \pi z$$

for all $z \in S(D^k)$ this implies that $\pi(S(D^k)) = \{0\}$, or, equivalently, that $S(D^k) \subseteq \ker(\pi) = P$ so that $Q_{P}D^k$ is continuous, a contradiction.

Therefore, without loss of generality, we may assume that for all $i \in \mathbb{N}$ we have $k_i \leq k$ and, since each $\mathcal{E}_q(D)$ is a finite set, that $\lim_{i \to \infty} \text{index}(P_i) = +\infty$. For each $i \in \mathbb{N}$ choose $m_i \in \{0, 1, 2, \ldots, k_i - 1\} \subseteq \{0, 1, 2, \ldots, k - 1\}$ such that

$$Q_{P_i}D^{m_i} : (\mathcal{A}, \|\cdot\|_{\text{index}(P_i)-1}) \to \mathcal{A}/P_i \cong \mathfrak{C}$$

is discontinuous. Since there are infinitely many $i$ but only finitely many choices for each $m_i$ we can drop to a subsequence of $\{\pi_i\}_{i=1}^{\infty}$ which uses $m_i = m \in \{0, 1, 2, \ldots, k - 1\}$ exclusively. That is, we can assume without loss of generality that there is $m \in \{0, 1, 2, \ldots, k - 1\}$ such that

$$Q_{P_i}D^m : (\mathcal{A}, \|\cdot\|_h) \to \mathcal{A}/P_i \cong \mathfrak{C}$$

is eventually discontinuous as $i \to \infty$ for each fixed $h \in \mathbb{N}$ (although each $Q_{P_i}D^m$ is certainly continuous with respect to the Fréchet space topology generated by all seminorms).

Let $T_i = \pi_iD^m$ for $i \in \mathbb{N}$ and let $T = \pi D^m$. It is clear that $\lim_{i \to \infty} T_ix = Tx$ for each fixed $x \in \mathcal{A}$. Since $Q_{P}D^m$ is continuous if and only if $S(D^m) \subseteq P$ if and only if $\pi(S(D^m)) = \{0\}$ if and only if $\pi^m$ is continuous, and since a similar argument applies for each $Q_{P_i}D^m$, it is clear that $T$ and $\{T_i\}_{i=1}^{\infty}$ are continuous linear operators from the Fréchet space $\mathcal{A}$ to the Banach space $\mathfrak{C}$. The previous lemma implies that there exists $p \in \mathbb{N}$ and $C \in \mathbb{R}^+$ such that

$$\|T_ix\| = |\pi_iD^m x| \leq C\|x\|_p$$

for all $x \in \mathcal{A}$ and $i \in \mathbb{N}$. But then

$$Q_{P_i}D^m : (\mathcal{A}, \|\cdot\|_p) \to \mathcal{A}/P_i \cong \mathfrak{C}$$

is continuous for all $i \in \mathbb{N}$, a contradiction. Hence no such sequence $\{\pi_n\}_{i=1}^{\infty}$ exists, ending the proof of the lemma.  

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An immediate corollary is

**Lemma 3.17.** Let $D$ be a (possibly discontinuous) derivation on a \emph{commutative} Fréchet algebra $\mathcal{A}$ whose structure space $\Phi_\mathcal{A}$ is a compact metric space in the weak* topology. Then the union of the exceptional sets $\bigcup_{q=1}^{\infty} \mathcal{E}_q(D)$ is a finite set and there exists $q_0$ such that $\bigcup_{q=1}^{\infty} \mathcal{E}_q(D) = \mathcal{E}_{q_0}(D)$. 
§4. Derivations on Radical Fréchet Algebras with Identity Adjoined.  
The results in the last section showed that the discontinuity, as measured by the separating subspaces
\[ S(D^k) \text{ for } k = 1, 2, 3, \ldots, \]
is associated with an non-decreasing sequence of finite subsets \( \{ \mathcal{E}_q(D) \}_{q=1}^{\infty} \) of strictly irreducible representations, each of whose kernels fails to contain some \( S(D^k) \).

Thus, it is natural to ask what the structure of \( D \) is like in the simplest possible (non-trivial) case, namely a derivation \( D \) on a commutative radical Fréchet algebra \( \mathcal{R}^\mathbb{R} \cong \mathbb{C} \cdot 1 \oplus \mathcal{R} \) with identity adjoined. Here \( \text{rad}(\mathcal{R}^\mathbb{R}) = \mathcal{R} \), all elements \( r \in \mathcal{R} \) have spectrum equal to \( \{0\} \), and there is precisely one continuous strictly irreducible representation \( \pi \in \text{Irred}(\mathcal{R}) \) taking \( (\lambda 1 + r) \) to \( \lambda \).

We are, of course, interested in the case where \( \mathcal{R} \) is not invariant under \( D \), and hence, by [Thomas2], \( \mathcal{R}^\mathbb{R} \) cannot be a Banach algebra. In addition, since \( \mathcal{R}^\mathbb{R} \) is commutative we can multiply \( D \) by a suitable invertible element in order to obtain a new derivation which takes some element \( x \) in the radical, \( \mathcal{R} \), to the identity, 1, in \( \mathcal{R}^\mathbb{R} \).

Recalling Definition 1.9, there is always a largest \( x \)-divisible subspace of \( \mathcal{R} \) which we will denote \( D_x \) and which is easily seen to lie in \( \mathcal{R} \). By [Thomas2, Lemma 2.8] the torsion subspace is \( x \)-divisible so that \( D_x = \bigcap_{m=1}^{\infty} x^m \mathcal{R} \). Since \( \sigma(x) = \{0\} \), and since the difference, \( (D(x \cdot) - xD(\cdot)) \), is continuous, we can apply a basic result from the theory of automatic continuity on Fréchet spaces [Thomas1, Lemma 1.2] to conclude that for every \( k \in \mathbb{N} \) there exists \( n(k) \in \mathbb{N} \) such that
\[ \overline{x^m S(D)^k} = \overline{x^{n(k)} S(D)^k} \subseteq \overline{D_x}^k \]
whenever \( m \geq n(k) \). With this as motivation, we make the following definition.

**Definition 4.1.** With the commutative radical Fréchet algebra \( \mathcal{R} \), the (possibly discontinuous) derivation \( D \), and the element \( x \in \mathcal{R} \) as above, we define for each \( k \in \mathbb{N} \) the set
\[ \mathcal{E}_k \equiv \{ a \in \mathcal{R} \mid x^n a \in \overline{D_x}^k \text{ for some } n \in \mathbb{N} \} \]
and we let \( \mathcal{E} \) denote the intersection of the closures:
\[ \mathcal{E} \equiv \cap_{k=1}^{\infty} \overline{\mathcal{E}_k} \]

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It is routine to check that $\{E_k\}_{k=1}^{\infty}$ is a non-increasing sequence of (not necessarily closed) ideals of $R^k$ and that

$$E_k \supseteq D_{x^k} = \cap_{m=1}^{\infty} x^m R^k$$

If we let $Q_k$ be the canonical continuous quotient map from $R^k$ onto $R^k / D_{x^k}$ it is clear that $Q_k x^{n(k)} S(D) = \{0\}$. Therefore $Q_k x^{n(k)} D$ is continuous, and for each $k \in N$ we can find $\ell(k) \in N$, larger than both $n(k)$ (so that $Q_k x^{\ell(k)} D$ is also continuous) and $k$ (so that $D_{x^k}^{\ell(k)} \subseteq D_{x^k}$), and $C_k > 0$ such that

$$\|Q_k x^{n(k)} D y\|_k \leq C_k \|y\|_{\ell(k)} \quad (4.2)$$

Since $D(D_{x^k}) \subseteq D_{x^k}$ it must also be true that

$$Q_k x^{n(k)} D(D_{x^k}) \subseteq Q_k x^{n(k)} D_{x^k} = \{0\}. \quad \text{Therefore, from inequality (4.2), we see that}$$

$$Q_k x^{n(k)} D(D_{x}^{\ell(k)}) = \{0\} \quad (4.3)$$

for $k = 1, 2, \ldots$. Let $a \in E_{\ell(k)}$ so that there exists $n \in N$ with $x^n a \in D_{x}^{\ell(k)}$. The Leibniz property shows that

$$Q_k x^{n+n(k)} D a = Q_k x^{n(k)} D(x^n a) - Q_k (n x^{n(k)+n-1} a) = 0 - 0 = 0$$

using equation (4.3), and the fact that $D_{x}^{\ell(k)} \subseteq D_{x^k}$. This shows that $D E_{\ell(k)} \subseteq E_k$ for each $k \in N$. Also, since $x^{n(k)} S(D) \subseteq D_{x}^{k}$ we have that $S(D) \subseteq E_k$ for each $k \in N$. For each $k \in N$ let $R_k$ be the canonical continuous quotient map from $R^k$ onto $R^k / E_k$. It is easily checked that $R_k S(D) = \{0\}$ and hence $R_k D$ is continuous for each $k \in N$. Since $R_k D(E_{\ell(k)}) = \{0\}$, it follows that $R_k D(E_{\ell(k)}) = \{0\}$. This shows that $D(E_{\ell(k)}) \subseteq E_k$ for all $k \in N$. We are now ready to state our first lemma.

**Lemma 4.4.** With the commutative radical Fréchet algebra $R^k$, the (possibly discontinuous) derivation $D$, and the element $x \in R$ as above, the set $E$ is a closed ideal of $R^k$ with the following properties:

(i.) $E$ is $D$-invariant.

(ii.) the separating ideal $S(D)$ is contained in $E$.

(iii.) $D$ drops to a continuous derivation $\hat{D}$ from the quotient Fréchet algebra $R^k / E$ to itself and $S(D^k) \subseteq E$ for all $k \in N$.

**Proof.** It is clear that $E$ is a closed ideal since each of the $E_k$'s is an ideal. Assertion (i.) follows from the fact that $D(E_{\ell(k)}) \subseteq E_k$ for each $k \in N$. Assertion
(ii.) follows from the fact that $S(D) \subseteq E_k$ for each $k \in N$. If $R$ is the canonical continuous quotient map from $\mathcal{R}^d$ onto $\mathcal{R}^{d}/E$ then $RS(D) = \{0\}$ and hence $RD$ is continuous. Consequently, since $RD(E) = \{0\}$, it must be the case that $RD$ factors through $\mathcal{R}^{d}/E$ as $DR$ where $D$ is a continuous derivation from $\mathcal{R}^{d}/E$ into itself defined by $D(a + E) = (Da + E)$ for $a \in \mathcal{R}^{d}$. This shows the final assertion, since $S(D^k) \subseteq E$ if and only if $RD^k$ is continuous, but $RD^k = D^kR$ for $k = 2, 3, \ldots$ and the maps on the right hand side are all continuous.

At this point we are confronted with three possibilities (however, we will see later that option (ii.) of Proposition 4.5 cannot occur). Reconsider Definitions 1.4 and 1.13 with regard to $\mathcal{R}^{d}/\bigcap_{m=1}^{\infty} x^m R$.

**Proposition 4.5.** Let $\mathcal{R}^{d}$ be a commutative radical Fréchet algebra with identity adjoined (so that $\mathcal{R}^{d} \cong \mathfrak{C} \cdot 1 \oplus \mathcal{R}$ and $\mathcal{R} = \text{rad}(\mathcal{R}^{d})$). Let $D$ be a (possibly discontinuous) derivation from $\mathcal{R}^{d}$ to itself and let $x \in \mathcal{R}$ satisfy $Dx = 1$. Then at least one of the following occurs:

(i.) $S(D^k) \subseteq \mathcal{R}$ for all $k \in N$ and, if $\mathcal{R}$ is a Banach algebra, then $D(\mathcal{R}) \subseteq \mathcal{R}$.

(ii.) For some $k_0 \in N$ we have $S(D^{k_0}) \not\subseteq \mathcal{R}$, $x$ is non-locally nilpotent, and $x$ has finite closed descent in $\mathcal{R}$, i.e. there exists $m_0 \in N$ such that

$$x^{m_0} \in \frac{\mathcal{R}^{d}}{\bigcap_{m=1}^{\infty} x^m \mathcal{R}} = \mathcal{D}_x$$

(iii.) For some $k_0 \in N$ we have $S(D^{k_0}) \not\subseteq \mathcal{R}$, and if $x$ denotes the coset $x = x + \bigcap_{m=1}^{\infty} x^m \mathcal{R}$ in $\mathcal{R}^{d}/\bigcap_{m=1}^{\infty} x^m \mathcal{R}$, then $x$ is locally nilpotent.

**Proof.** Suppose that assertion (i.) holds and that $\mathcal{R}$ is a Banach algebra. Since $\mathcal{R}$ is a primitive ideal of $\mathcal{R}^{d}$ of finite codimension, Lemma 3.4 assertion (iv.) shows that $D(\mathcal{R}) \subseteq \mathcal{R}$.

Now assume that $S(D^{k_0}) \not\subseteq \mathcal{R}$ for some $k_0 \in N$. Apply Lemma 4.4 assertion (iii.). Since $S(D^k) \subseteq E$ for all $k \in N$ it cannot be the case that $E \subseteq \mathcal{R}$. But then $E$ contains an invertible element, and, being a closed ideal, must contain the identity element, 1. This means that $1 \in \mathcal{E}_k$ for all $k \in N$. But if $\mathcal{E}_k \subseteq \mathcal{R}$ then $\overline{\mathcal{E}_k} \subseteq \mathcal{R}$ since the radical, $\mathcal{R}$, is closed. It must therefore be the case that $1 \in \mathcal{E}_k$ for all $k \in N$. Hence, for each $k \in N$ there exists $m(k) \in N$ such that

$$x^{m(k)} \in \mathcal{D}_x^k$$

(4.6)
At this point we consider the coset \( \hat{x} = x + \overline{ \cap_{m=1}^{\infty} x^m R } \) in the quotient Fréchet algebra \( \mathcal{R}^d / \overline{ \cap_{m=1}^{\infty} x^m R } \). If \( x \) is locally nilpotent, so is \( \hat{x} \) and assertion (iii.) holds. If \( x \) is non-locally nilpotent but \( \hat{x} \) is nilpotent then \( x^{m_0} \in \overline{ \cap_{m=1}^{\infty} x^m R } \) for some \( m_0 \in \mathbb{N} \). Since \( D_x = \cap_{m=1}^{\infty} x^m R \) the fact that

\[
x^{m_0} \in \overline{D_x} \subseteq \overline{x^{m_0+1} R^d} \subseteq \overline{ \cap_{m=1}^{\infty} x^m R }
\]

shows that assertion (ii.) holds.

Finally suppose that \( x \) is non-locally nilpotent and \( \hat{x} \) is non-nilpotent. In this case, statement (4.6) as well as the fact that \( D_x = \cap_{m=1}^{\infty} x^m R \), shows that for each \( k \in \mathbb{N} \) we have

\[
\| x^{m(k)} + \cap_{m=1}^{\infty} x^m R \|_k = 0
\]

This proves that \( \hat{x} \) is locally nilpotent and establishes assertion (iii.)  

The paper [Thomas2] (which handles the Banach space case) outlines necessary modifications (on page 450) for Fréchet spaces, but is rather sketchy and does not demonstrate that case (ii.) of Proposition 4.5 cannot occur. We give the full argument below.

First note that \( \mathcal{R}^d / D_x = \mathcal{R}^d / \cap_{m=1}^{\infty} x^m R \) has a natural \( p \)-adic topology \( \tau_x \) with local base at zero consisting of the sets of cosets

\[
\{ x^k R + \cap_{m=1}^{\infty} x^m R \}_{k=1}^\infty
\]

It should be emphasized that \( \tau_x \) is definitely not a Fréchet space topology. The surprising fact is that finite closed descent (Proposition 4.5, case (ii.)), or local nilpotency (Proposition 4.5, case (iii.)), both imply statement (4.6) and this is sufficient to make the topology \( \tau_x \) complete.

**Lemma 4.7.** Let \( \mathcal{R}^d \) be a commutative radical Fréchet algebra with identity adjoined (so that \( \mathcal{R}^d \cong \mathcal{C} \cdot 1 \oplus \mathcal{R} \) and \( \mathcal{R} = \text{rad}(\mathcal{R}^d) \)). Let \( x \in \mathcal{R} \) and satisfy \( D_x = \cap_{m=1}^{\infty} x^m R \). Suppose that for every \( i \in \mathbb{N} \) there is \( m(i) \in \mathbb{N} \) such that

\[
x^{m(i)} \in \overline{D_x^d},
\]

(this holds in cases (ii.) and (iii.) of Proposition 4.5). Let \( \{a_i\}_{i=1}^\infty \) be any sequence of coefficients chosen from \( \mathcal{R}^d \) to form the formal series \( \sum_{i=m_1}^\infty a_k x^k \). Then there
exists an element $y_1 \in \overline{D}_{x}^{1}$ satisfying

$$y_1 - \left( \sum_{k=m_1}^{(m_{n_1}+1)(n-1)} a_k x^k \right) \in x^n \overline{D}_{x}^{n+1}$$

for $n = 1, 2, 3, \ldots$. Consequently, letting a “dot” denote the coset in the quotient Fréchet algebra $\mathcal{R}^g / \cap_{m=1}^{\infty} \mathcal{R}$, the series $\sum_{k=m_1}^{\infty} \hat{a}_k \hat{x}^k$ converges to $\hat{y}_1$ in the natural $p$-adic topology $\tau_x$ on $\mathcal{R}^g / \cap_{m=1}^{\infty} x^m \mathcal{R}$. Consequently, every absolutely summable series converges and $(\mathcal{R}^g / \mathcal{D}_x, \tau_x)$ is complete.

**Proof.** Note that each closed ideal $\overline{D}_x^n$ is a Fréchet space in the relative topology. For each $n \in \mathbb{N}$ define the affine maps $A_n : \overline{D}_x^{n+1} \to \overline{D}_x^n$ as follows:

$$A_1 y = xy + \sum_{k=m_1}^{m_2} a_k x^k \quad \text{(recall that } x^{m_1} \in \overline{D}_x^1)$$

$$A_2 y = xy + \sum_{k=m_2+1}^{m_3+1} a_k x^k \quad \text{(recall that } x^{m_2} \in \overline{D}_x^2)$$

$$A_3 y = xy + \sum_{k=m_3+1}^{m_4+2} a_k x^k \quad \text{(recall that } x^{m_3} \in \overline{D}_x^3)$$

$$\vdots$$

$$A_n y = xy + \sum_{k=m_{n-1}+1}^{m_n+(n-1)} a_k x^k \quad \text{(recall that } x^{m_{n-1}} \in \overline{D}_x^n)$$

Note that $\sum_{m_{n-1}+1}^{m_n+(n-1)} a_k x^k \in \overline{D}_x^n$ for $n = 1, 2, \ldots$. Since $x \mathcal{D}_x = \mathcal{D}_x$ we have that

$$\overline{A_n \mathcal{D}_x^{n+1}} \supset \overline{A_n \mathcal{D}_x^n} = \overline{\mathcal{D}_x^n}$$

for $n = 1, 2, \ldots$. In addition, since each $A_n$ is affine we have that

$$\|A_n a - A_n b\|_{n+p} = \|x(a - b)\|_{n+p} \leq \|x\|_{n+p} \|a - b\|_{n+p}$$

for all $a, b \in \overline{\mathcal{D}_x^{n+1}}$ and $p = 0, 1, 2, \ldots$.

Lemma 2.2 above, with $C_i = \|x\|_i$ and $X_n = \mathcal{D}_x^n$, shows that the inverse, or projective, limit $P = \{ (y_n)_{n=1}^{\infty} \mid A_n y_{n+1} = y_n \text{ for } n = 1, 2, \ldots \}$ is non-empty and that $\pi_n(P)_n = \mathcal{D}_x^n$ (where $\pi_n$ is the $n$-th coordinate projection) for $n = 1, 2, \ldots$. So, let $(y_n)_{n=1}^{\infty} \in P$ and compute:

$$y_1 = A_1 y_2 = xy_2 + \sum_{k=m_1}^{m_2} a_k x^k$$
\[ y_2 = A_2 y_3 = x y_3 + \sum_{k=m_2+1}^{m_3+1} a_k x^{k-1} \quad \text{and hence} \quad y_1 = x^2 y_3 + \sum_{k=m_1}^{m_3+1} a_k x^k \]

\[ y_3 = A_3 y_4 = x y_4 + \sum_{k=m_3+2}^{m_4+2} a_k x^{k-2} \quad \text{and hence} \quad y_1 = x^3 y_4 + \sum_{k=m_1}^{m_4+2} a_k x^k \]

\[ \ldots \]

\[ y_n = A_n y_{n+1} = x y_{n+1} + \sum_{k=m_n+(n-1)}^{m_{n+1}+(n-1)} a_k x^{k-(n-1)} \quad \text{and hence} \quad y_1 = x^n y_{n+1} + \sum_{k=m_1}^{m_{n+1}+(n-1)} a_k x^k \]

This shows that

\[ y_1 - \left( \sum_{k=m_1}^{m_{n+1}+(n-1)} a_k x^k \right) \in x^n D_x^{-n+1} \]

for \( n = 1, 2, \ldots \). Note that if we had started with a different element \((y'_1)_{n=1}^\infty \in P\) we would have that \((y_1 - y'_1) \in \cap_{m=1}^\infty x^m R\) and so \( y_1 = y'_1 \) in \( R^d / \cap_{m=1}^\infty x^m R \). Therefore, the series \( \sum_{k=m_1}^{\infty} a_k \hat{x}^k \) converges to \( y_1 \) in the natural \( p \)-adic topology \( \tau_x \) on \( R^d / \cap_{m=1}^\infty x^m R \). The fact that completeness is equivalent to every absolutely summable series converging is well known. This finishes the proof of the lemma. \( \blacksquare \)

From this point on we can follow either [Thomas2, Proposition 2.18 to Proposition 2.24] or [Zariski, Lemma 4] almost verbatim to establish our main result.

**Theorem 4.8.** Let \( R^d \) be a commutative radical Fréchet algebra with identity adjoined (so that \( R^d \cong \mathbf{C} \cdot 1 \oplus R \) and \( R = \text{rad}(R^d) \)). Let \( D \) be a (possibly discontinuous) derivation from \( R^d \) to itself and let \( x \in R \) be any element with \( Dx \) invertible in \( R^d \). Then at least one of the following occurs:

(i.) \( S(D^k) \subseteq R \) for all \( k \in \mathbb{N} \) and, if \( R \) is a Banach algebra, then \( D(R) \subseteq R \).

(ii.) \( R^d \) has a formal power series quotient based at \( x \), \( x \) does not have finite closed descent, \( R \) is not a Banach algebra, and if \( \hat{x} \) denotes the coset containing \( x \) in the quotient Fréchet algebra \( R^d / \cap_{m=1}^\infty x^m R \), then \( \hat{x} \) is locally nilpotent (but non-nilpotent).

**Proof.** Since \( R^d \) is commutative and \( Dx = u \) with \( u \) invertible in \( R^d \) we can replace \( D \) by the derivation \( \tilde{D} \) (where \( \tilde{D} \) is defined by \( \tilde{D}(v) = u^{-1} D(u) \)). Note
then that $\bar{D}(x) = 1$. We will now drop the “tilde” and write $D$ for $\bar{D}$. Apply Proposition 4.5. If assertion (i.) holds we are done.

Now assume either assertion (ii.) or assertion (iii.) holds. Since statement (4.6) is then true, we have that $({\mathcal{R}}^d/\mathcal{D}_x, \tau_x)$ is complete as a consequence of Lemma 4.7. For $\hat{x} \in {\mathcal{R}}^d/\mathcal{D}_x$ define

$$\theta(\hat{a}) = \sum_{k=0}^{\infty} \left( (-1)^k \frac{D^k(a) x^k}{k!} + \mathcal{D}_x \right),$$

the series converging as a consequence of the completeness. It is easily checked that $\theta$ is both a unital algebra homomorphism and projection of $\mathcal{R}^d/\mathcal{D}_x$ onto a unital subalgebra $\mathcal{A}_0$ of $\mathcal{R}^d/\mathcal{D}_x = \mathcal{R}^d/\cap_{m=1}^{\infty} x^m \mathcal{R}$. In addition, the kernel of $\theta$ is precisely $\hat{x}(\mathcal{R}^d/\mathcal{D}_x)$, and $\bar{D} \circ \theta = 0$. We have already remarked that the derivation $D$ leaves $\mathcal{D}_x$ invariant so it drops to a derivation $\hat{D}$ from $\mathcal{R}^d/\mathcal{D}_x$ to itself. One then defines an algebra homomorphism

$$\hat{a} \rightarrow \sum_{n=0}^{\infty} \left( \frac{\theta(D^n(a))}{n!} x^n \right)$$

from $\mathcal{R}^d/\mathcal{D}_x$ into $\mathcal{A}_0[[X]]$. It is clear that this homomorphism is onto. The fact that the kernel of $\theta$ is $\hat{x}(\mathcal{R}^d/\mathcal{D}_x)$, together with $Dx^n = nx^{n-1}$ for all $n \in \mathbb{N}$ show that the homomorphism is injective.

Finally, apply Proposition 2.6(vii.) and the fact the $\mathcal{R}^d$ is a commutative radical Fréchet algebra with identity adjoined to show that the first alternative is not possible and that the second alternative must hold (that is, $x$ cannot have finite closed descent, $\mathcal{R}$ is not a Banach algebra, and $\hat{x}$ is locally nilpotent but non-nilpotent). This demonstrates that assertion (iii.) rather than assertion (ii.) of Proposition 4.5 must have held, and finishes the justification of option (ii.) of our current theorem.

At one time the author believed that condition (i.) of Theorem 4.8 was true for both Fréchet algebras and Banach algebras. A recent example in [Read2] shows that there exists a radical Fréchet algebra $\mathcal{R}$ of formal power series with a locally nilpotent indeterminate $X$ which has a discontinuous derivation $D$ on its unitization $\mathcal{R}^d$ such that $D(X) = 1$ with the separating ideal $\mathcal{S}(D)$ equal to the entire algebra $\mathcal{R}^d$. This example shows that condition (i.) may fail for a Fréchet algebra, and hence condition (ii.) is not vacuous.
References


