Mathematics 222 – Module 6
Continuation of Ordinary Differential Equations of
the Form: \( y'(x) = F(y(x), x) \) and Investigation
of Steady State Solutions

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Sometimes we want the plots (or graphs) Maple V is drawing to be in a
separate window. To do this, after pulling in the worksheet, set Options–>
Plot Display–>Window. When you print such a plot it will be on its own
(single) page.

In this first example we have a resistor of \( R \) Ohms in series with a capacitor
of \( C \) Farads which we hook up to a signal generator which puts out an
impressed voltage \( E(t) = \sin(\omega t) \) which is a one volt peak-to-peak sine
wave of frequency \( \omega \). The same current \( I(t) \) must flow through all circuit
components and Kirchoff’s law says that the voltage across the resistor \( E_1(t) \)
plus the voltage across the capacitor \( E_2(t) \) must add up to \( E(t) \). In addition
we have Ohm’s law, which must be satisfied by the resistor,

\[
E_1(t) = RI(t)
\]

and another equation which must be satisfied by the capacitor

\[
\frac{dE_2(t)}{dt} = \frac{I(t)}{C}
\]

since capacitance opposes a change in voltage and this opposition (as mea-
sured by the rate of change of voltage) is inversely proportional to the
size of the capacitor. If we solve these two equations for \( I(t) \) and replace
\( E_1(t) = E(t) - E_2(t) \) we get

\[
C \frac{dE_2(t)}{dt} = \frac{E(t) - E_2(t)}{R}
\]

After we move \( C \) to the right hand side, this will be our \texttt{ode1} (with \( y(t) =
E_2(t) \)) for the voltage (see below). We can also write down an ODE for the
current \( I(t) \) by noting that

\[
\frac{dI(t)}{dt} = \frac{1}{R} \frac{dE_1(t)}{dt} = \frac{1}{R} \left( \omega \cos(\omega t) - \frac{dE_2(t)}{dt} \right)
\]
This will be our ode2 (with \( y(t) = I(t) \)) for the current (see below). We will solve both these ODE’s and plot sufficiently many solutions to see whether there is any trend of the solutions to approach a steady state.

Start by loading the following packages and functions

```maple
with(student);
with(plots);
readlib(unassign);
```

Set the impressed voltage to be a one volt peak-to-peak sine wave with frequency \( \pi \) as follows

```maple
E := t -> sin(w*t);
w := Pi;
```

Then set up and solve the differential equation for the voltage

```maple
ode1 := diff(y(t),t) = (E(t) - y(t))/(R1*C1);
```

or (with pre-version 5.5)

```maple
ode1 := dsolve(ode1);
```

Note the complexity of the general solution but also note that some of the terms will go to zero as \( t \to \infty \) (why?). We want to define about 30 solutions for different values of constants \( C1 \), and we also need to set the resistor and capacitor values. This can be done rather easily with the following loop:

```maple
for i from -15 to 15 do:
v[i](t) := subs(C1=i*exp(-abs(i/2)),R1=1,
            C=1,rhs(h1)):
end;
```

You should not copy this statement blindly but examine the syntax so that you understand what is being done as Maple goes through the loop 31 times.

Note that the voltage solution corresponding to \( C1 = 0 \) is of the form

\[
\frac{1}{\sqrt{1 + \omega^2 R1^2 C1^2}} \left( \frac{-\omega R1 C1}{\sqrt{1 + \omega^2 R1^2 C1^2}} \cos(\omega t) + \frac{1}{\sqrt{1 + \omega^2 R1^2 C1^2}} \sin(\omega t) \right)
\]

Verify this by doing

```maple
simplify(v[0](t));
```

Next set up and solve the differential equation for the current

```maple
ode2 := diff(y(t),t) = (diff(E(t),t) - (y(t)/C1))/R1;
h2 := dsolve(ode2);
```

or (with pre-version 5.5)
\[ h2 := \text{dsolve}(\text{ode2}, y(t)); \]
\[ \text{for } i \text{ from } -15 \text{ to } 15 \text{ do:} \]
\[ c[i](t) := \text{subs}(_C1=i*\exp(-\text{abs}(i/2)), R1=1, \]
\[ _C1=1, \text{rhs}(h2)); \]
\[ \text{od;} \]

Note that the current solution corresponding to \(_C1 = 0\) is of the form
\[
\frac{\omega C1}{\sqrt{1 + \omega^2 R1^2 C1^2}} \left( \frac{1}{\sqrt{1 + \omega^2 R1^2 C1^2}} \cos(\omega t) + \frac{\omega R1 C1}{\sqrt{1 + \omega^2 R1^2 C1^2}} \sin(\omega t) \right) (**)
\]
Verify this by doing
\[ \text{simplify}(c[0](t)); \]

We want to plot the impressed voltage, \(E(t)\), the solutions for negative constants, the solution for a zero constant, and the solutions for positive constants on the same graph, first for the voltage solutions, then for the current solutions.

Make these plots in separate windows, print them and label them (respectively) “Voltage Solutions for \(\omega = \pi\)” and “Current Solutions for \(\omega = \pi\)” and turn these in with your assignment below. This can be done with:
\[ \text{plot}([E(t), \text{seq}(v[i](t), i=-15..15)], t=-18/\pi..18/\pi, \]
\[ y=-1.5..1.5, \text{resolution}=1000, \text{thickness}=[3, \]
\[ \text{seq}(0, i=-15..1), 3, \text{seq}(0, i=1..15)], \text{color}=[\text{red}, \]
\[ \text{seq}(\text{black}, i=-15..1), \text{blue}, \text{seq}(\text{black}, i=1..15)], \]
\[ \text{linestyle}=[0, \text{seq}(0, i=-15..15)]; \]
\[ \text{plot}([E(t), \text{seq}(c[i](t), i=-15..15)], t=-18/\pi..18/\pi, \]
\[ y=-1.5..1.5, \text{resolution}=1000, \text{thickness}=[2, \]
\[ \text{seq}(0, i=-15..1), 3, \text{seq}(0, i=1..15)], \text{color}=[\text{red}, \]
\[ \text{seq}(\text{black}, i=-15..1), \text{blue}, \text{seq}(\text{black}, i=1..15)], \]
\[ \text{linestyle}=[0, \text{seq}(0, i=-15..15)]; \]

Examine both plots carefully. Note that all of the voltage curves approach the solution with \(_C1 = 0\) as \(t \to \infty\). Note that the current curves do the same. Also note that there is a slight phase shift of the steady state current wave when compared to the impressed voltage. We can compute this phase shift from the solutions of \text{ode2} to be
\[ \text{phase angle } = \cos^{-1}(\omega R/\sqrt{(1/C)^2 + \omega^2 R^2}) \]
\[
= \cos^{-1}\left(\frac{\omega RC}{\sqrt{1+\omega^2(\omega^2C^2)}}\right)
\]

Compute this with Maple as follows:

```maple
phase := subs(R1=1,C1=1,arccos((w*R1)/
sqrt((1/C1^2) + w^2*R1^2)));
evalf(phase);
```

**Assignment:**

1. Do the same analysis as you did above but with the **lower** frequency \(\omega = \pi/4\), that is, go back and set

   \[w := \frac{\pi}{4};\]

   rerun the commands and make the two plots in separate windows, print them and label them (respectively) “Voltage Solutions for \(\omega = \pi/4\)” and “Current Solutions for \(\omega = \pi/4\)” and turn these in with your assignment.

2. Do the same analysis as you did above but with the **higher** frequency \(\omega = 4\pi\), that is, go back and set

   \[w := 4\pi;\]

   rerun the commands and make the two plots in separate windows, print them and label them (respectively) “Voltage Solutions for \(\omega = 4\pi\)” and “Current Solutions for \(\omega = 4\pi\)” and turn these in with your assignment.

Analyze the 3 sets of voltage and current solutions (6 plots total) and do the following:

3. Identify the impressed voltage \(E(t)\) on all 6 plots and label clearly as “Impressed Voltage,” identify the steady state voltage solution (*) on the 3 voltage plots and label clearly as “Steady State Voltage,” and identify the steady state current solution (**) on the 3 current plots and label clearly as “Steady State Current.”

4. Assume that the resistor and capacitor values are fixed. Is the voltage across the capacitor greatest when the frequency \(\omega\) is large or small. Explain why by referring to (*).

5. Assume that the resistor and capacitor values are fixed. Is the current flowing in the circuit greatest when the frequency \(\omega\) is large or small. Explain why by referring to (**).